# Mereology and Metricality

*Formal Appendix*

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This is the formal appendix to the paper titled "Mereology and Metricality". It comprises two parts. The first part, below, deals with the proving a representation and uniqueness theorem about the Mereological-Reductive account of volume presented in the main text (and glossed below). The second part, starting on page 44 with section B, deals with other interesting results, including some lemmas important to the proofs in the first part.

## The M-R Account of Volume

Definitions:

**Concatenation/"Put together in the right way":** $ab \circ c =_{df}$ $a$ and $b$ don't overlap, and compose $c$.

**Definition of $\leq$:** $a \leq b =_{df}$ $a$ bears $\approx$ to some part of $b$.

**Definition of $\prec$:** $a \prec b =_{df}$ $a$ bears $\approx$ to some part of $b$ and $a \not\approx b$.

**Definition of "Sum":** $c$ is as voluminous as $a$ and $b$ put together (alternatively, $c$'s volume is the sum of $a$'s and $b$'s volumes) $=_{df}$ there exists an $x \approx a$ and $y \approx b$ such that $xy \circ c$

Axioms:

(**V-Comb**): If $a$ and $b$ are voluminous, don't overlap, and compose $c$, then $c$ is voluminous. (i.e. if $ab \circ c$, then $c \approx c$)

(**Additivity**): If $ab \circ c$, then, for any $x \approx c$, if $x$ has parts $y$ and $z$ which don't overlap and compose it, then $y \approx a$ just in case $z \approx b$.

(**Properly Extensive**): If $a$ bears $\approx$ to a part of $b$ and $b \approx c$, then either $a \approx b$ or there exists an object, $x \approx x$, such that $ax \circ c$.

(**Totality**): If $a$ and $b$ are voluminous, then either $a$ bears $\approx$ to some part of $b$ or $b$ bears $\approx$ to some part of $a$.

Simplifying Assumption:

**Within-Object Archimedean Property:** If $a$ bears $\approx$ to a part of $b$, then every set, $S$, of non-overlapping parts of $b$ such that $\forall x(x \in S \rightarrow x \approx a)$, is finite.

This system also presupposes the axioms of classical extensional mereology (CEM).

(Note: In the remainder of this paper, voluminous entities denoted by primed terms always bear $\approx$ to the denotation of the corresponding unprimed term. So $x \approx x'$, for any voluminous $x$.)

# A The Direct Ratio Theorem

First, I'll specify a condition which all and only the acceptable functions, $\varphi$, from voluminous objects to the positive reals must satisfy. We'll write "Taking $a$ out of $b$ outputs a count of $k$ and a remainder of $r$" as "$T.O.(a,b,k,r)$". Here is the condition on $\varphi$:[43]

**RULE:** *If* ($T.O.(a,b,k,r)$ or $T.O.(a,b,k,\emptyset)$) *then* $\varphi(b) = k * \varphi(a) + n$ where $k \in \mathbb{Z}$ and $n = \varphi(r)$ if there exists a remainder and 0 otherwise.

From this rule alone, it's very straightforward to prove what I call the *Direct Ratio Theorem*. It is a variant of the more familiar style of representation and uniqueness theorems. I call it a "ratio theorem" because it explicitly asserts the correspondence between the Vrat relation and the mathematical ratio between the numbers any $\varphi$ assigns to voluminous objects (I use '$\psi : A \mapsto B$' to denote a function, $\psi$, from set $A$ to set $B$).

**Direct Ratio Theorem.** *Every function $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**) if and only if, for all $a, b \in V$.*

*(i)* $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

*(ii)* Vrat:$n(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

*Moreover, for any pair of function $\varphi$ and $\varphi'$ which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such that, for all $x \in V$:*

$$\varphi(x) = m * \varphi'(x)$$

*Where $m$ is such that, if there exists some $u, v \in V$ where $\varphi(v) = \varphi'(u)$, then* Vrat:$m(u,v)$.

## A.1 Preliminary Lemmas

### A.1.1 An Important Result

**Lemma 2.** *If $a \prec b$, where $a, b \in V$, then, for any $\varphi : V \mapsto \mathbb{R}^+$ that satisfies (**RULE**), $\frac{\varphi(b)}{\varphi(a)} > 1$.*[44]

[43] I will sometimes, for simplicity, refer instead to the weaker:

**RULE***: *If $T.O.(a,b,k,r)$ then* $\varphi(b) = k * \varphi(a) + \varphi(r)$ (where $k \in \mathbb{Z}^+ \cup 0$ and $r \preceq a$).

Suppressing the case where there is no remainder, (since that is just the special case where $a \approx b$) the simplified version (**RULE***) is equivalent to the more complicated (**RULE**) when it is combined with the stipulation that $a \approx b \Leftrightarrow \varphi(a) = \varphi(b)$.

[44] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, 59.

*Proof.* Pick some arbitrary $a, b \in \mathbf{V}$ and an arbitrary function, $\varphi$, that satisfies (**RULE**). Suppose $a \prec b$. By (**RULE**), then, $\varphi(b) = k * \varphi(a) + \varphi(r)$. So $\frac{\varphi(b)}{\varphi(a)} = k + \frac{\varphi(r)}{\varphi(a)}$. Since its numerator and denominator are positive reals, the fraction $\frac{\varphi(r)}{\varphi(a)} \in \mathbb{R}^+$, and, since $a \prec b$, $k \geq 1$. So their sum is $> 1$.

□

### A.1.2 The Key Lemma

The next step is that we prove the following:[45]

**Lemma 3.** *For any voluminous $a$ and $b$, and any $\varphi : \mathbf{V} \mapsto \mathbb{R}^+$ that satisfies (**RULE**), $K(a,b) = \langle k_0, k_1, k_2, \ldots \rangle$ just in case*

$$\frac{\varphi(b)}{\varphi(a)} = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \ddots}}}$$

*That is,*

$$K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle \text{ iff } \frac{\varphi(b)}{\varphi(a)} = k_0 + \frac{1}{k_1 +} \frac{1}{k_2 +} \frac{1}{k_3 +} \cdots \tag{1}$$

*Proof.* Suppose that $\varphi$ is a function from $\mathbf{V}$ to $\mathbb{R}^+$ that satisfies (**RULE**).

Consider some arbitrary voluminous pair $a, b \in \mathbf{V}$.

By (**RULE**), $x \approx y$ if and only if $\varphi(x) = \varphi(y)$. So, if $a \approx b$, then $K(a,b) = \langle 1 \rangle$ (by step 0 of the ratio procedure) and $\frac{\varphi(b)}{\varphi(a)} = 1$, since $\varphi(a) = \varphi(b)$.

Now assume $a \not\approx b$. Then there exists some

$$K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle$$

If there exists a $\frac{\varphi(b)}{\varphi(a)}$, then, by the arguments in Appendix section (B.5), $\frac{\varphi(b)}{\varphi(a)}$ is a real number with a unique simple continued fraction expansion (where the final denominator, if it exists, is $\geq 2$). That is, there exists some non-negative integers $f_0, f_1, f_2, \dots \in \mathbb{Z}$ such that

$$\frac{\varphi(b)}{\varphi(a)} = f_0 + \frac{1}{f_1 +} \frac{1}{f_2 +} \frac{1}{f_3 +} \cdots \tag{2}$$

[45] The proof of this lemma directly depends on Lemma 2. For the full map of Lemma interdependence, see figure 4, p. 59.

It suffices to show that for every $i$ such that $k_i$ is defined, there exists an $f_i$ such that $k_i = f_i$. We demonstrate this below:

**1.**

We perform the ratio procedure for $a$ and $b$. The first step of this procedure is to take $a$ out of $b$. This will output a count, $s \in \mathbb{Z}$, and a remainder which we'll call "$r_0$". By (**RULE**),

$$\varphi(b) = s * \varphi(a) + \varphi(r_0) \tag{3}$$

Either $r_0 \approx a$ or not. We reason by cases. First, suppose $r_0 \approx a$, then (by the definition of the ratio procedure) $k_0 = s + 1$. Also, by (**RULE**), $\varphi(r_0) = \varphi(a)$. Hence

$$\varphi(b) = (s+1) * \varphi(a) \tag{4}$$

Dividing both sides by $\varphi(a)$ yields

$$\frac{\varphi(b)}{\varphi(a)} = \frac{(s+1) * \varphi(a)}{\varphi(a)} \tag{5}$$
$$\frac{\varphi(b)}{\varphi(a)} = f_0 = s + 1$$

So, if $r_0 \approx a$, then $K(a, b)$ has one member and $f_0 = k_0 = s + 1$. Now suppose $r_0 \not\approx a$. Then $k_0 = s$. So

$$\varphi(b) = s * \varphi(a) + \varphi(r_0) \tag{6}$$

Dividing both sides by $\varphi(a)$ yields

$$\begin{aligned} \frac{\varphi(b)}{\varphi(a)} &= \frac{s * \varphi(a) + \varphi(r_0)}{\varphi(a)} \\ &= s + \frac{\varphi(r_0)}{\varphi(a)} \\ &= s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} \end{aligned} \tag{7}$$

$T.0.(a, b, s, r_0)$ just in case $r_0 \preceq a$. Since $r_0 \not\approx a$, then $r_0 \prec a$. By Lemma 2, $\frac{\varphi(a)}{\varphi(r_0)} > 1$. Hence, $s$ is the greatest integer less than $\frac{\varphi(b)}{\varphi(a)}$. So $s = f_0$. So in this case, as well, there exists a $k_0$ and $k_0 = f_0$. Since $k_0 = f_0$ in both cases, $k_i = f_i$ for $i = 0$.

2.

We proceed to step 2 of the ratio procedure and take $r_0$ out of $a$. Hence, when we take $r_0$ out of $a$, the procedure will output a count, $t \in \mathbb{Z}^+$, and a remainder $r_1 \leq r_0$. By (**RULE**),

$$\varphi(a) = t * \varphi(r_0) + \varphi(r_1) \tag{8}$$

Either $r_1 \approx r_0$ or not. We reason by cases. First, suppose $r_1 \approx r_0$. So $k_1 = t + 1$, and $K(a,b) = \langle s, t+1 \rangle$. Also by (**RULE**), $\varphi(r_1) = \varphi(r_0)$. Hence

$$\begin{aligned} \varphi(a) &= t * \varphi(r_0) + \varphi(r_0) \\ \varphi(a) &= (t+1) * \varphi(r_0) \end{aligned} \tag{9}$$

Dividing both sides by $\varphi(r_0)$ yields..

$$\frac{\varphi(a)}{\varphi(r_0)} = \frac{(t+1) * \varphi(r_0)}{\varphi(r_0)} \tag{10}$$

$$\frac{\varphi(a)}{\varphi(r_0)} = t + 1 \tag{11}$$

Which means that

$$\begin{aligned} \frac{\varphi(b)}{\varphi(a)} &= s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} = s + \frac{1}{t+1} \\ \frac{\varphi(b)}{\varphi(a)} &= f_0 + \frac{1}{f_1} = k_0 + \frac{1}{k_1} = s + \frac{1}{t+1} \end{aligned} \tag{12}$$

(Since $t \neq 0$, $t+1 \geq 1$. Hence $k_0 + \frac{1}{k_1}$ is a simple continued fraction whose final denominator is $\geq 2$.) So, if $r_1 \approx r_0$, then $k_i = f_i$ for $i \leq 1$. Now suppose $r_1 \not\approx r_0$. Then $k_1 = t$ and $K(a,b) = \langle s, t, \ldots \rangle$. Recall,

$$\varphi(a) = t * \varphi(r_0) + \varphi(r_1) \tag{13}$$

Dividing both sides by $\varphi(r_0)$ yields..

$$\frac{\varphi(a)}{\varphi(r_0)} = \frac{t * \varphi(r_0) + \varphi(r_1)}{\varphi(r_0)}$$

$$\frac{\varphi(a)}{\varphi(r_0)} = t + \frac{\varphi(r_1)}{\varphi(r_0)} = t + \cfrac{1}{\cfrac{\varphi(r_0)}{\varphi(r_1)}} \tag{14}$$

By Lemma 2, $\frac{\varphi(r_0)}{\varphi(r_1)} > 1$, so the inverse is $< 1$, meaning $t$ is the greatest integer less than $\frac{\varphi(a)}{\varphi(r_0)}$. Hence,

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} \\ &= s + \cfrac{1}{t + \cfrac{\varphi(r_1)}{\varphi(r_0)}} \\ \frac{\varphi(b)}{\varphi(a)} &= s + \cfrac{1}{t + \cfrac{1}{\cfrac{\varphi(r_0)}{\varphi(r_1)}}}\end{aligned} \tag{15}$$

So in this case, as well, there exists a $k_0$ and a $k_1$ $k_0 = f_0 = s$ and $k_1 = f_1 = t$. Since $k_1 = f_1$ in both cases, $k_i = f_i$ for all $i \leq 1$.

Since $r_1 \not\approx r_0$, then $r_1 \prec r_0$. We proceed to step 3 and take $r_1$ out of $r_0$. This procedure will output a count $u \in \mathbb{Z}$, and a remainder $r_2 \preceq r_1$... (and so on).

**In the general case:**

Suppose that $k_i = f_i$ for any $i \leq n-2$. Suppose that $K(a,b)$ has at least $n-1$ members (i.e. $k_{n-2}$ is not $K(a,b)$'s final member). We show that $k_{n-1} = f_{n-1}$.

By the definition of the ratio procedure, there exits remainders $r_j$ for all $j \leq n-1$, and $r_{n-1} \not\approx r_{n-2}$. Since $r_{n-1} \not\approx r_{n-2}$, we proceed to step $N$ and take $r_{n-1}$ out of $r_{n-2}$. This procedure will output a count, $v \in \mathbb{Z}$, and a remainder $r_n \preceq r_{n-1}$. By (**RULE**),

$$\varphi(r_{n-2}) = v * \varphi(r_{n-1}) + \varphi(r_n) \tag{16}$$

Either $r_n \approx r_{n-1}$ or not. We reason by cases. First, suppose $r_n \approx r_{n-1}$, then $k_{n-1} = v+1$. Also, this means $\varphi(r_n) = \varphi(r_{n-1})$. Hence

$$\begin{aligned}\varphi(r_{n-2}) &= v * \varphi(r_{n-1}) + \varphi(r_n) \\ &= v * \varphi(r_{n-1}) + \varphi(r_{n-1}) \\ &= (v+1) * \varphi(r_{n-1})\end{aligned} \tag{17}$$

Dividing both sides by $\varphi(r_{n-1})$ yields..

$$\begin{aligned}\frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= \frac{(v+1) * \varphi(r_{n-1})}{\varphi(r_{n-1})} \\ &= v+1\end{aligned} \tag{18}$$

Which means that $K(a,b) = \langle s, t, u, \ldots, v+1 \rangle$ and

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= s + \frac{1}{t+}\,\frac{1}{u+} \cdots \frac{1}{\dfrac{\varphi(r_{n-2})}{\varphi(r_{n-1})}} \\ &= s + \frac{1}{t+}\,\frac{1}{u+} \cdots \frac{1}{v+1}\end{aligned} \tag{19}$$

So, if $r_n \approx r_{n-1}$, then $K(a,b)$ has $n-1$ members and $\frac{\varphi(b)}{\varphi(a)} = f_0 + \frac{1}{f_1+}\,\frac{1}{f_2+} \cdots \frac{1}{f_{n-1}} = k_0 + \frac{1}{k_1+}\,\frac{1}{k_2+} \cdots \frac{1}{k_{n-1}}$. Now suppose $r_{n-1} \not\approx r_{n-2}$. Then $k_{n-1} = v$. So

$$\varphi(r_{n-2}) = v * \varphi(r_{n-1}) + \varphi(r_n) \tag{20}$$

Dividing both sides by $\varphi(r_{n-1})$ yields..

$$\begin{aligned}\frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= \frac{v * \varphi(r_{n-1}) + \varphi(r_n)}{\varphi(r_{n-1})} \\ &= v + \frac{\varphi(r_n)}{\varphi(r_{n-1})} \\ \frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= v + \frac{1}{\dfrac{\varphi(r_{n-1})}{\varphi(r_n)}}\end{aligned} \tag{21}$$

By Lemma 2, $\frac{\varphi(r_{n-1})}{\varphi(r_n)} > 1$, so the inverse is $< 1$, meaning $v$ is the greatest integer less than $\frac{\varphi(n-2)}{\varphi(r_{n-1})}$.

Hence,

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= s + \frac{1}{t+}\,\frac{1}{u+}\cdots\frac{1}{\dfrac{\varphi(r_{n-2})}{\varphi(r_{n-1})}} \\ &= s + \frac{1}{t+}\,\frac{1}{u+}\cdots\frac{1}{v+}\,\frac{1}{\dfrac{\varphi(r_{n-1})}{\varphi(r_n)}}\end{aligned} \tag{22}$$

So in this case, as well, there exists a $k_{n-1}$ and $k_{n-1} = f_{n-1} = v$. Since $k_{n-1} = f_{n-1}$ in both cases, $k_i = f_i$ for all $i \leq n-1$.

By induction on these steps, we can conclude that for any $i$, if $k_i$ exists then $f_i$ exists and $k_i = f_i$.

Since (1) is satisfied for an arbitrary voluminous pair $a, b$ and an arbitrary function, $\varphi$, from voluminous entities to positive real numbers that satisfies (**RULE**), it follows that (1) is satisfied for any choice of $a$ and $b$ and for any $\varphi$ satisfying (**RULE**). □

We have shown that Lemma 3 holds. Hence, for any voluminous $a \preceq b$ and any function $\varphi$ from voluminous objects to real numbers that satisfies (**RULE**),

$$K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots\rangle \text{ iff } \frac{\varphi(b)}{\varphi(a)} = k_0 + \frac{1}{k_1+}\,\frac{1}{k_2+}\,\frac{1}{k_3+}\cdots \tag{1}$$

Here is how we can proceed from Lemma 3 to the Direct Ratio Theorem. Recall that the definition of the various relations VRAT:$n(b,a)$ for any $a \preceq b$ was simply that $n = k_0 + \frac{1}{k_1+}\,\frac{1}{k_2+}\,\frac{1}{k_3+}\ldots$, where $K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots\rangle$. Hence Lemma 3 implies that, for any voluminous $a$ and $b$, $\frac{\varphi(b)}{\varphi(a)} = n$, where $n$ is the real number corresponding to the volume ratio relation such that VRAT:$n\, b, a$.

## A.2 Proof of the Direct Ratio Theorem

Recall that the Direct Ratio Theorem says:

**Direct Ratio Theorem.** *Every function $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**) if and only if, for all $a, b \in V$.*

*(i) $a \prec b$ iff $\varphi(a) \leq \varphi(b)$*

*(ii)* VRAT:$n(b,a)$ *iff $\varphi(b) = n * \varphi(a)$*

*Moreover, for any pair of function $\varphi$ and $\varphi'$ which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such* *that, for all $x \in V$:*

$$\varphi(x) = m * \varphi'(x)$$

*Where $m$ is such that, if there exists some $u, v \in V$ where $\varphi(v) = \varphi'(u)$, then* Vrat:$m(u,v)$.

*Proof.* We'll divide this theorem up into four lemmas, and prove each of those, occasionally relying on the result obtained in the proof of the previous lemma.

**Lemma 4.** *If $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**), then it satisfies*

*(i)* $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

*(ii)* Vrat:$n(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

*for all $a, b \in V$.*[46]

From Lemma 3, for any voluminous pair $a$ and $b$, $\frac{\varphi(b)}{\varphi(a)} = n$, where $n$ is the real number corresponding to the volume ratio relation such that Vrat:$n(b,a)$.

Given this result, the proof for (ii) is trivial:

*Proof.* By Lemma 3, for any voluminous pair $a$ and $b$, $\frac{\varphi(b)}{\varphi(a)} = n$ if and only if $n$ is the real number corresponding to the volume ratio relation such that Vrat:$n(b,a)$. Since $\frac{\varphi(b)}{\varphi(a)} = n$ just in case $\varphi(b) = n * \varphi(a)$ (multiplying both sides by $\varphi(a)$), we can conclude that Vrat:$n(b,a)$ iff $\varphi(b) = n * \varphi(a)$. □

The proof for (i) is also quite simple:

*Proof.* Taking each direction of the biconditional in turn:

**Left to Right:** By the definition of the ratio relation, if Vrat:$n(x,y)$, then $n > 1$ just in case $y \prec x$. Because, the definition of the ratio relation says that $n = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\ldots$ where $k_0, k_1, \ldots$ are the first, second, etc. elements of the list $K(y,x)$. And, by the definition of the ratio procedure, $k_0 \neq 0$ and $k_1$ is defined iff $y \prec x$ (if $y \approx x$ then $k_1$ is not defined, and if $x \prec y$ then $k_0 = 0$).

Hence, for an arbitrary $a, b \in \mathbf{V}$, if $a \prec b$ then the $n$ such that Vrat:$n(b,a)$ is $> 1$. By Lemma 3, $\frac{\varphi(b)}{\varphi(a)} > 1$, since $\frac{\varphi(b)}{\varphi(a)} = n$. So $\varphi(b) > \varphi(a)$.

**Right to Left:** We prove the contrapositive. For some arbitrary $a, b \in \mathbf{V}$. If $\frac{\varphi(b)}{\varphi(a)} \leq 1$, then, by Lemma 3, it couldn't be that Vrat:$n(b,a)$ for $n > 1$. Hence, by the definition of the ratio relation, $\neg(a \prec b)$.

□

**Lemma 5.** *If $\varphi : V \mapsto \mathbb{R}^+$ satisfies*

[46] The proof of this lemma directly depends on Lemma 3. For the full map of Lemma interdependence, see figure 4, p. 59.

*(i)* $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

*(ii)* VRAT:$n(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

*for all* $a, b \in \mathbf{V}$*, then* $\varphi$ *satisfies (**RULE**).*[47]

*Proof.* Let $\varphi$ be some function from voluminous entities to positive real numbers such that (i) and (ii). It suffices to show that $\varphi$ satisfies:

**RULE:** *If* $T.O.(a,b) \Rightarrow \langle k, d_k \rangle$ *then* $\varphi(b) = k * \varphi(a) + \varphi(d_k)$ (where $k \in \mathbb{Z}$ and $d_k \preceq a$).

Suppose $T.O.(a,b) \Rightarrow \langle k, d_k \rangle$. It follows that $a \preceq b$, so (by the ratio procedure) $K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle$ is defined. Hence (by the definition of the ratio relations) VRAT:$n(b,a)$, where $n = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\ldots$. By (ii),

$$\frac{\varphi(b)}{\varphi(a)} = n = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\ldots$$

By the definition of the ratio procedure, $k_0$ is either the count, $c \in \mathbb{Z}$, output by taking $a$ out of $b$, or $k_0 = c+1$, in which case $a \approx r_0$. We reason by cases. Suppose $k_0 = c+1$, hence $a \approx r_0$. By the definition of the ratio procedure, this means that $k_0$ is $K(a,b)$'s first and only member. By (ii), this means that

$$\frac{\varphi(b)}{\varphi(a)} = k_0 = c+1$$

Hence

$$\varphi(b) = (c+1) * \varphi(a) = c * \varphi(a) + 1 * \varphi(a)$$

Since $a \approx r_0$ if and only if $\varphi(a) = \varphi(r_0)$, then $\varphi(b) = c * \varphi(a) + \varphi(r_0)$. Hence, if $k_0 = c+1$, then $\varphi$ satisfies (**RULE**).

Now suppose $k_0 = c$. Hence $r_0 \prec a$ or the remainder is null. If the remainder is null, then $c = 1$ and $a \approx b$. Hence $\varphi(a) = \varphi(b)$, so $\varphi(b) = 1 * \varphi(a)$. If $r_0 \prec a$, then $K(a,b)$ has at least two members.

Consider the list $K(a,b)$ and the list $K(r_0,a)$. It is clear from the definition of the ratio procedure that, the $(i+1)$'th element in $K(a,b)$ is identical to the $i$'th element of $K(r_0,a)$. That is, the members of $K(a,b)$, in order, starting with $k_1$ (i.e. excluding $k_0$), is identical to the ordered list of integers, $K(r_0,a)$.

[47] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

So, labeling the elements of $K(a,b)$ as '$k_i$' and the elements of $K(r_0,a)$ as $f_i$, by (ii) it follows that:

$$\frac{\varphi(b)}{\varphi(a)} = n = k_0 + \frac{1}{k_1+}\dots$$

and

$$\frac{\varphi(a)}{\varphi(r_0)} = f_0 + \dots = k_1 + \dots$$

Hence

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= k_0 + \frac{1}{\frac{\varphi(a)}{\varphi(r_0)}}\\ &= k_0 + \frac{\varphi(r_0)}{\varphi(a)}\end{aligned}$$

Multiplying both sides of the equation by $\varphi(a)$ yields:

$$\begin{aligned}\varphi(b) &= (k_0 + \frac{\varphi(r_0)}{\varphi(a)}) * \varphi(a)\\ &= k_0 * \varphi(a) + (\frac{\varphi(r_0)}{\varphi(a)}) * \varphi(a)\\ \varphi(b) &= k_0 * \varphi(a) + \varphi(r_0)\end{aligned}$$

Hence, if $r_0 \prec a$, then $\varphi$ satisfies (**RULE**).

Since it can be shown for all cases, it follows that $\varphi$ satisfies (**RULE**). Since we reached this conclusion from the assumption that $\varphi$ satisfies (i) and (ii), it follows that any function $\varphi$ which satisfies (i) and (ii) satisfies (**RULE**). □

**Lemma 6.** *If $\varphi$ and $\psi$ are both functions from* **V** *to the non-negative reals which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such that, for all $x \in V$:*[48]

$$\varphi(x) = m * \psi(x)$$

*Proof.* Suppose there exists some $\varphi$ and $\psi$, both functions from the domain of voluminous objects to the reals. Suppose $\varphi$ and $\psi$ satisfy (**RULE**). Hence, by Lemma 4, $\varphi$ and $\psi$ satisfy (i) and (ii).

We want to show that, for all $x \in \mathbf{V}$, $\varphi(x) = m * \psi(x)$ for some $m$. Note that, since $\varphi(x) \in \mathbb{R}$

[48]The proof of this lemma directly depends on Lemmas 4 and 11. For the full map of Lemma interdependence, see figure 4, p. 59.

and $\psi(x) \in \mathbb{R}$, then clearly $\frac{\varphi(x)}{\psi(x)} \in \mathbb{R}$. As such, it will suffice to show that, for every $u, v \in \mathbf{V}$, $\frac{\varphi(u)}{\psi(u)} = \frac{\varphi(v)}{\psi(v)}$.

Pick an arbitrary voluminous pair, $a, b \in \mathbf{V}$. By Lemma 11, for every ordered pair there exists some ratio relation, call it 'Vrat:$n$', such that Vrat:$n(a,b)$. Since both $\varphi$ and $\psi$ satisfy (ii),

$$\varphi(a) = n * \varphi(b) \tag{23}$$

and

$$\psi(a) = n * \psi(b) \tag{24}$$

Hence,

$$\begin{aligned}\frac{\varphi(a)}{\psi(a)} &= \frac{n * \varphi(b)}{n * \psi(b)} \\ &= \frac{n}{n} * \frac{\varphi(b)}{\psi(b)}\end{aligned}$$

therefore

$$\frac{\varphi(a)}{\psi(a)} = \frac{\varphi(b)}{\psi(b)}$$

We have shown that $\frac{\varphi(a)}{\psi(a)} = \frac{\varphi(b)}{\psi(b)}$. Since it is true for an arbitrary $a, b \in \mathbf{V}$ it holds for any voluminous pair. Since $\frac{\varphi(a)}{\psi(a)} \in \mathbb{R}^+$, then there exists some $m \in \mathbb{R}^+$ such that, for any $x \in \mathbf{V}$, $\frac{\varphi(x)}{\psi(x)} = m$, i.e. $\varphi(x) = m * \psi(x)$. □

Finally, we show that,

**Lemma 7.** *Suppose that $\varphi$ and $\psi$ are functions from $V$ to the non-negative reals such that*

$$\varphi(x) = m * \psi(x)$$

*For all $x \in V$, then, if there exists some $u, v$ such that $\varphi(v) = \psi(u)$, then $m$ is the real number such that* Vrat:$m(u,v)$.[49]

*Proof.* Given Lemma 6, It suffices to show that, for some $u, v, x \in \mathbf{V}$, if Vrat:$n(u,v)$ and $\varphi(x) = m * \psi(x)$, then $m = n$.

[49]The proof of this lemma directly depends on Lemmas 6. For the full map of Lemma interdependence, see figure 4, p. 59.

Suppose that $\textsc{Vrat}{:}n(u,v)$ and $\varphi(x) = m * \psi(x)$ for some $x \in \mathbf{V}$. By Lemma 6, $\varphi(x) = m * \psi(x)$ for *any* $x \in \mathbf{V}$. Since $u \in \mathbf{V}$, it follows that:

$$\frac{\varphi(u)}{\psi(u)} = m$$

Moreover, from (ii), we know that

$$\varphi(u) = n * \varphi(v)$$

Hence

$$n = \frac{\varphi(u)}{\varphi(v)}$$

However, since $\varphi(v) = \psi(u)$, it follows that

$$n = \frac{\varphi(u)}{\psi(u)}$$

Hence $n = \frac{\varphi(u)}{\psi(u)} = m$. So from the assumption that $\varphi(v) = \psi(u)$, it follows that $\textsc{Vrat}{:}m(u,v)$ (where $m = \frac{\varphi(x)}{\psi(x)}$ for any $x \in \mathbf{V}$). □

By Lemmas 4 and 5, every function $\varphi$ from $\mathbf{V}$ to the non-negative real numbers satisfies (**RULE**) if and only if, for all $a, b \in \mathbf{V}$.

$$a \prec b \;\leftrightarrow\; \varphi(a) \leq \varphi(b)$$

and

$$\textsc{Vrat}{:}n(b,a) \;\leftrightarrow\; \varphi(b) = n * \varphi(a)$$

And, from Lemmas 6 and 7, if $\varphi$ and $\varphi'$ are functions from $\mathbf{V}$ to the non-negative reals which satisfy (**RULE**), then there exists some $m \in \mathbb{R}^+$ such that, for all $x \in \mathbf{V}$:

$$\varphi(x) \;=\; m * \varphi'(x)$$

Where $m$ is such that, if there exists some $u, v \in \mathbf{V}$ where $\varphi(v) = \varphi'(u)$, then $\textsc{Vrat}{:}m(u,v)$.

Therefore, Lemmas 4, 5, 6, and 7 entail the Direct Ratio Theorem. □

# B Additional Results and Useful Lemmas

This section contains proofs of various lemmas, some of which were explicitly appealed to in the main text or in the proof of the Direct Ratio Theorem, others of which are of independent interest. Sections B.1 and B.2 show that the two procedures presented in sections 4.1 and 4.2.1 of the main text are, indeed, functions from pairs of voluminous objects to their respective outputs. Section B.3 outlines a few lemmas appealed to in proofs elsewhere in the paper (in the main text as well as other parts of this appendix). Section B.4 connects the axioms I took to characterize proper extensiveness in section 3.6 in the main text to my presentation of them in Perry (2015). Section B.5 provides a discussion of continued fractions in case the reader is unfamiliar, and shows how each real numbers has can be uniquely expressed as a simple continued fraction.

## B.1 The Taking-out Procedure is a Function

First we'll show, in the case of the taking-out procedure, that the output of that procedure is *defined* for any ordered pair of voluminous entities, $\langle a, b\rangle$, and is *unique* up to the volume of the remainder. Then we'll show that, given this result, the output of the ratio procedure is defined and unique for any ordered pair of voluminous entities.

**Lemma 8.** *For any voluminous pair $a, b$, taking $a$ out of $b$ always has an output, either $\{1\}$ or $\{k, r\}$ were $k \in \mathbb{Z}$ and $r \preceq a$ is a part of $b$.*[50]

The definition of the taking-out procedure, presented in Section 4.1, as well as the argument presented footnote 34 of the main text, amounts to a proof of this lemma.

**Lemma 9.** *The taking-out-of procedure for a voluminous pair is unique up to the volume of the remainder (that is, different remainders are possible for the same pair just in case they have the same volume).*[51]

*Proof.* Consider an arbitrary $a \approx a$ and $b \approx b$. We show that the procedure for taking $a$ out of $b$ is unique up to the volume of the remainder:

Uniqueness for the special cases—that is, where either $b \prec a$ or $b \approx a$—is built in: By the definition of the taking-out-of procedure, taking $a$ out of $b$ outputs a count of 1 and no remainder if *and only if* $b \approx a$. Similarly, by the definition of the procedure, taking $a$ out of $b$ outputs a count of 0 and a remainder of $b$ if and only if $b \prec a$.

[50] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[51] The proof of this lemma directly depends on Lemmas 13, 14, and 15. For the full map of lemma interdependence, see figure 4, p. 59.

Since $b \prec a$, $a \approx b$, and $a \prec b$ are mutually exclusive (this follows from Lemma 14) and exhaustive possibilities for a voluminous pair, these biconditionals entail that the procedure for taking $a$ out of $b$ is unique for $a \approx b$ and $b \prec a$.

Now we show that the procedure is unique for $a \prec b$. Let $T.O.(a,b,n,r)$ be elliptical for "at least one performance of the procedure for taking $a$ out of $b$ results in a count of $n \in \mathbb{Z}$ and a remainder, $r$". Since $a \not\approx b$, there will always exist a remainder.

To prove uniqueness up to the volume of the remainder, it suffices to show that, for any voluminous pair $a \prec b$: If $T.O.(a,b,n,r)$, then, for any $m \in \mathbb{Z}$ and any voluminous part $r^*$ of $b$, $T.O.(a,b,m,r^*)$ if and only if $n = m$ and $r \approx r^*$.

First, we'll show that, if $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $n = m$. Then we'll show that, if $n = m$, then $r \approx r^*$

We show that: $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*) \Rightarrow n = m$

Suppose, for reductio, that $n \neq m$, then either $n > m$ or $n < m$.

Suppose WLOG that $n < m$.

By the definition of the "taking out" procedure, $n$ and $m$ are the respective cardinalities of sets, $S$ and $S^*$, of parts of $b$, where every member of the set bears $\approx$ to $a$, and no member of either set overlaps any other members of that set. $S^* = \{x_1, \ldots, x_m\}$. By Lemma 15, $fus(S) \approx fus(\{x_1,...,x_n\})$. (I will use the term '$[..\oplus..]$' to represent the fusion of a pair of objects, and '$fus(X)$' to represent the fusion of all members of the set, $X$)

Since $fus(S^*)$ doesn't overlap $r^*$, $fus(\{x_1,...,x_n\})$ and $r^*$ don't overlap. No members of $S^*$ overlap, so $fus(\{x_1,...,x_n\})$ and $fus(\{x_{n+1},...,x_m\})$ don't overlap. Hence, $fus(\{x_1,...,x_n\})$ and $[fus(\{x_{n+1},...,x_m\}) \oplus r^*]$ don't overlap and jointly compose $b$. By (Additivity), $fus(S) \circ r = b$ and $fus(S) \approx fus(\{x_1,...,x_n\})$ imply that $r \approx [fus(\{x_{n+1},...,x_m\}) \oplus r^*]$.

Consider the fusion of $x_{n+1} \in S^*$ and $r^*$, call it "$[x_{n+1} \oplus r^*]$". $[x_{n+1} \oplus r^*]$ is a part of $[fus(\{x_{n+1},...,x_m\}) \oplus r^*]$. Since $x_{n+1}$ and $r^*$ don't overlap, $x_{n+1} \circ r^* = [x_{n+1} \oplus r^*]$ which, by (V-Comb), entails that $[x_{n+1} \oplus r^*]$ is voluminous.

Since $[x_{n+1} \oplus r^*]$ bears $\approx$ to a part of $[fus(\{x_{n+1},...,x_m\}) \oplus r^*]$ (viz. itself), and $r \approx [fus(\{x_{n+1},...,x_m\}) \oplus r^*]$, by (Properly Extensive), $[x_{n+1} \oplus r^*]$ bears $\approx$ to a part of $r$. Call this part "$p$".

By (Properly Extensive) again, $x_{n+1}$ bears $\approx$ to a part, "$x'_{n+1}$", of $p$ (since $p \approx [x_{n+1} \oplus r^*]$). $p \leq r$. If $x'_{n+1}$ were $\approx r$, then, by (Properly Extensive), $x'_{n+1}$ would have a part $\approx p$. But, by Lemma 14, if $x'_{n+1} \prec p$, then there are no parts of $x'_{n+1}$ that bear $\approx$ to $p$. Hence $x'_{n+1} \not\approx r$.

But, since $x'_{n+1} \approx x_{n+1} \approx a$, $a$ bears $\approx$ to a part of $r$. And, since $x'_{n+1} \not\approx r$, $a \not\approx r$. So $a \prec r$. But $T.O.(a,b,n,r)$, which means $r$ is the remainder output of a performance of the "taking out of" procedure on $a$ and $b$. But, by the definition of that procedure, a remainder is output only when the procedure terminates, and the procedure terminates (for $a \prec b$) just in case the remainder is $\preceq a$. Therefore because $T.O.(a,b,n,r)$, $r \preceq a$. But if $a \prec r$, then $r$ cannot be $\preceq a$ (see Lemma 14). Contradiction!

Hence, if $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $n = m$.

Now we show that: $T.O.(a,b,n,r)$, $T.O.(a,b,m,r^*)$, and $n = m \Rightarrow r \approx r^*$

Suppose for reductio that $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ and $a \prec b$, but $r \not\approx r^*$.

By the previous argument, $n = m$. Let '$S$' denote the set of $n$ non-overlapping parts of $b$ such that every member of $S$ bears $\approx$ to $a$ and $fus(S) \circ r = b$, and '$S^*$' the set of $m$ non-overlapping parts of $b$ such that every member of $S^*$ bears $\approx$ to $a$ and $fus(S) \circ r^* = b$. By Lemma 15 and $n = m$, the fusion of all of $S$'s members bears $\approx$ to the fusion of all of $S^*$'s members. That is $fus(S) \approx fus(S^*)$.

Since we've assumed that $r \not\approx r^*$, by (Totality), either $r \prec r^*$ or $r^* \prec r$. Suppose (WLOG) that $r \prec r^*$. Then there exists a part, $x$, of $r^*$ such that $x \approx r$. By (Properly Extensive) and $x \not\approx r^*$, there must also exist a part, $y$, of $r^*$ such that $x \circ y = r^*$.

By (Totality), either $r \prec r^*$ or $r^* \prec r$. Suppose (WLOG) that $r \prec r^*$. Then there exists a part, $x$, of $r^*$ such that $x \approx r$. By (Properly Extensive) and $x \not\approx r^*$, there must also exist a part, $y$, of $r^*$ such that $x \circ y = r^*$.

$fus(S^*)$ doesn't overlap $r^*$ (or any of its parts). So $fus(S^*) \circ x = [fus(S^*) \oplus x]$ and, by (V-Comb), $[fus(S^*) \oplus x]$ is voluminous. By Lemma 13, from $fus(S) \approx fus(S^*)$, $r \approx x$, and $fus(S) \circ r = b$, it follows that $[fus(S^*) \oplus x] \approx b$.

However, since $x \circ y = r^*$, and $fus(S^*)$ and $r^*$ compose $b$, it follows that $[fus(S^*) \oplus x] \circ y = b$. But, by (Within-Object Archimedean), $[fus(S^*) \oplus x] \not\approx b$. Contradiction!

So, $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $r \approx r^*$. □

**Lemma 10.** *If $T.O.(a,b,n,r)$ and $a \approx c$ and $b \approx d$, then $T.O.(c,d,n,r')$, where $r' \approx r$.*[52]

*Proof.* In the case where $a \approx b$ (and, therefore, $c \approx d$), this is trivial, since $n = 1$ and there is no remainder in both instances of the taking-out procedure.

Suppose $a \not\approx b$

[52] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

**Case 1:** Taking $a$ out of $b$ has no remainder.

Suffices to show that: For all voluminous $x, y, z$, If $x$ is composed of $n$-many non-overlapping parts, all bearing $\approx$ to $y$, then $z \approx x$ is composed of $n$-many non-overlapping parts, all bearing $\approx$ to $y$.

> Suppose $x$ is composed of $n$-many non-overlapping parts all $\approx y$, $n \neq 1$. Let $S$ be the set of $n-1$ of those parts, and consider the fusion $fus(S)$. Call the $n$'th part (not in $S$) $y'$. Since $fus(S)$ and $y'$ don't overlap, $fus(S) \circ y' = x$.
>
> Since $z \approx x$, by (Properly Extensive), $z$ has a part bearing $\approx$ to $y'$. Call it $y''$. By ($\approx$ Trans), $z \not\approx y'$ so, by (Properly Extensive), there exists some other part, $w$, of $z$ such that $w \circ y'' = z$. By (Additivity), it follows that $w \approx fus(S)$.
>
> We have shown that, if $x$ is composed of $n$-many non-overlapping parts all $\approx y$, then $z \approx x$ is composed of two non-overlapping parts: one $\approx y$, and one bearing $\approx$ to the fusion of $n-1$-many non-overlapping parts all $\approx y$. Induction on this result—plus (Within-Object Archimedean), which says that there are no voluminous fusions of infinitely many non-overlapping parts all $\approx y$—implies that $z$ is the fusion of $n$-many non-overlapping parts all $\approx y$.

From this result and the transitivity of $\approx$, Lemma 10 follows for the case where taking $a$ out of $b$ has no remainder.

**Case 2:** Taking $a$ out of $b$ has a remainder, $r$.

> Suppose $T.O.(a, b, n, r)$. Let $S$ be a particular set of $n$ non-overlapping parts of $b$ all bearing $\approx$ to $a$. It follows that $fus(S)$ is a part of $b$, and that $fus(S) \approx fus(S)$. By (Properly Extensive), since $b \approx d$, $d$ has a proper part $x$ such that $x \approx fus(S)$.
>
> By the definition of the procedure, $(fus(S), r) \circ (b)$. By (Within-Object Archimedean), $b \not\approx fus(S)$, hence $d \not\approx fus(S)$ so $d \not\approx x$. By (Properly Extensive), there exists a $y$ such that $(x, y) \circ (d)$. By (Additivity), $x \approx fus(S)$ and $(fus(S), r) \circ (b)$ imply that $y \approx r$.
>
> By the definition of the procedure: $r \leq a$ and, by (Properly Extensive), $r \leq c$ since $a \approx c$. Since $d$ can be partitioned into a pair of non-overlapping parts, one of which is the fusion of $n$-many non-overlapping parts all $\approx c$ and the other is a remainder $y \approx r$, then $T.O.(c, d, n, y)$.
>
> Therefore, $T.O.(c, d, n, y)$, where $y \approx r$, follows from the assumption that $T.O.(a, b, n, r)$, $a \approx c$, and $b \approx d$. So in the case where taking $a$ out of $b$ has a remainder, the conditional in 10.

Since it holds in the only two cases, it holds in general. □

What function can we associate with the taking-out procedure? The procedure takes two voluminous entities as inputs and outputs an integer and, usually, another voluminous entity. However, Lemmas 9 and 10 show that only these entities' volumes matter to the procedure.

So the function we associate with this procedure should map pairs of volume properties to pairs of integers and volume properties. That is, if $T.O.(a,b,n,r)$, and $V_x(a)$, $V_y(b)$, and $V_z(r)$, then the function should map $\langle V_x, V_y \rangle$ to $\{n, V_z\}$. We can define an analogous function in terms of '$\approx$' by replacing the volume properties with equivalence classes. In that case, the function maps pairs of equivalence classes, $\{x \mid x \approx a\}$ and $\{x \mid x \approx b\}$, to integer-equivalence class pairs, $n$ and $\{x \mid x \approx r\}$.

## B.2 The Ratio Procedure is a Function

Turning now to the ratio procedure. We show that the ratio procedure has a defined output for any pair (Lemma 11) and we show that that output is unique up to the volume of that pair (Lemma 12).

**Lemma 11.** *For any ordered pair of voluminous objects $\langle a,b \rangle$, there exists a list of non-negative integers $K(a,b) = \langle k_0, k_1, k_2, \ldots \rangle$ constructable via the ratio procedure on $\langle a,b \rangle$.*[53]

Given the existence of an output of the taking-out procedure for any ordered pair $a$ and $b$ Lemma 8, the definition of the ratio procedure in section 4.2.1 amounts to a proof of the existence of a $K(a,b)$ for any $a$ and $b$.

**Lemma 12.** *If $K(a,b)$ is the list of integers output by the ratio procedure for a given $\langle a,b \rangle$, where $a$ and $b$ are voluminous, then, for any other ordered pair of voluminous objects $\langle c,d \rangle$ such that $a \approx c$ and $b \approx d$, there exists a list of non-negative integers, $K(c,d)$, constructable via the ratio procedure on $\langle c,d \rangle$ and $K(c,d) = K(a,b)$.*[54]

*Proof.* In addition to $K(a,b)$, the ratio procedure on $\langle a,b \rangle$ supplies us with a (possibly infinite) list of successive remainders, $r_i$, including an $r_{final}$ just in case the procedure terminates. Each of these $r_i$'s are parts of either $a$ or $b$. Specifically, if $i = 2j$ for some $j \in \mathbb{Z}$, then $r_i$ is part of $a$, and, if $i = 2j + 1$, $r_i$ is part of $b$.

Consider a token application of the ratio procedure on $\langle a,b \rangle$, where the list $R = \langle \ldots, r_i, r_{i+1}, \ldots \rangle$ of the successive remainders, denoted by '$r_i$' for various $i \in \mathbb{Z}$. Recall that each remainder, $r_i$, is a particular voluminous object, that is a part of either $a$ or $b$.

Also consider an application of the ratio procedure on $\langle c,d \rangle$, for which an entirely different list of voluminous objects, $R^* = \langle \ldots, r_i^*, r_{i+1}^*, \ldots \rangle$ are the successive remainders. Here the various $r_i^*$'s are either parts of $c$ or parts of $d$.

[53] The proof of this lemma directly depends on Lemma 8. For the full map of Lemma interdependence, see figure 4, p. 59.

[54] The proof of this lemma directly depends on Lemmas 9 and 10. For the full map of Lemma interdependence, see figure 4, p. 59.

**Step 1:** Show that, for all $i$, $r_i \approx r_i^*$.

At the first step this is trivial: $r_0$ and $r_{-1}$ are just $a$ and $b$, respectively. Likewise $r_0^* = c$ and $r_{-1}^* = d$. By assumption $a \approx c$ and $b \approx d$, so $r_i \approx r_i^*$ for $i = -1$ and $i = 0$.

By Lemma 10, if $T.O.(r_n, r_{n-1}, x, r_{n+1})$ for some $x \in \mathbb{Z}$, then for any $r_n^* \approx r_n$ and $r_{n-1}^* \approx r_{n-1}$, it follows that $T.O.(r_n^*, r_{n-1}^*, x, r_{n+1}^*)$ and $r_{n+1}^* \approx r_{n+1}$. By Lemma 9, the $T.O.$ procedure is unique up to the volume of the remainder, so *any* part $y$ of $d$ such that $T.O.(r_n^*, r_{n-1}^*, x, y)$ must be $\approx r_{n+1}^*$ and, therefore, $\approx r_{n+1}$.

So, if $r_n \approx r_n^*$ and $r_{n-1} \approx r_{n-1}^*$, then $r_{n+1} \approx r_{n+1}^*$ so long as $r_{n+1}$ and $r_{n+1}^*$ are defined. Hence, since $r_{-1} \approx r_{-1}^*$ and $r_0 \approx r_0^*$, by induction it follows that $r_i \approx r_i^*$ for all $i \in \mathbb{Z}^+$ for which $r_i$ exists.

**Step 2:** Show that the count output at each step is unique.

By the argument in Step 1, for any remainders $r_i$ and $r_i^*$ of two applications of the ratio procedure on $\langle a, b\rangle$ and $\langle c, d\rangle$, respectively, $r_i \approx r_i^*$. By Lemma 10, if taking $r_i$ out of $r_{i-1}$ outputs a count of $c \in \mathbb{Z}$, then taking $r_i^*$ out of $r_{i-1}^*$ will output a count of $c$ if $r_i \approx r_i^*$ and $r_{i-1} \approx r_{i-1}^*$.

Hence, by this and Lemma 9 the count output at every step of the ratio procedure is unique, no matter which parts are involved in the procedure.

Consider the list $K(a,b)$ output by the ratio procedure on $\langle a, b\rangle$. Each entry, $k_i$, in this list is either the count output by taking some remainder $r_i$ out of the previous remainder $r_{i-1}$, or (in the case of $K(a,b)$'s last entry, if it has one) it is that count +1. The value of $k_i$ in $K(a,b)$, therefore, depends on the count output at step $i+1$, and on whether the procedure terminates at that step.

**Step 3:** Show that the ratio procedure on $\langle a, b\rangle$ terminate at the $i$'th step if and only if the ratio procedure on $\langle c, d\rangle$ does.

An application of the ratio procedure terminates at the $i$-th step *if and only if* either (1) there's no remainder (only possible in first step) or if (2) the $i$'th remainder bears $\approx$ to the previous remainder.

(1) is trivial. $c \approx a$ and $d \approx b$. If taking $a$ out of $b$ has not remainder at the first step, then $a \approx b$. By ($\approx$ Trans) and ($\approx$ Sym), it follows that $c \approx d$ and, hence, taking $c$ out of $d$ has no remainder.

Regarding (2), it suffices to show that if $r_i, r_{i-1}, r_i^*$, and $r_{i-1}^*$ exist, then: $r_i \approx r_{i-1} \leftrightarrow r_i^* \approx r_{i-1}^*$.

Left to right: Suppose that these remainders exist and that $r_i \approx r_{i-1}$. By the argument in Step 1, for any remainder $r_i$ from the ratio procedure on $\langle a, b\rangle$ and any remainder $r_j^*$ from the ratio procedure on $\langle c, d\rangle$, if $i = j$ then $r_i \approx r_j^*$. Hence $r_i^* \approx r_i$ and $r_{i-1}^* \approx r_{i-1}$. By ($\approx$ Trans) and ($\approx$ Sym), it follows that $r_i^*$

By parallel reasoning, we can show the right to left direction of the biconditional.

Therefore, $k_i \in \mathbb{Z}$ is $K(a,b)$'s final entry if and only if $k_i^*$ is $K(c,d)$'s final entry.

By the argument in Step 2, if neither $k_i$ and $k_i^*$ are their list's final entry, then $k_i = k_i^*$. Likewise, since the final entry is just the count +1, if both $k_i$ and $k_i^*$ *are* their list's final entry, then $k_i = k_i^*$.

Since all of their entries are the same, $K(a,b) = K(c,d)$. □

Hence, the ratio procedure associates an ordered pair of volume properties (or equivalence classes of voluminous entities) with a *unique* list of non-negative integers. For a given ordered pair of voluminous entities, it outputs the list of integers associated with the ordered pair of volume properties that those two, respectively, instantiate.

### B.3 Other Useful Lemmas

This section contains three lemmas useful in simplifying proofs elsewhere in the paper. Lemma 13 is the "Unique Sum Lemma" in the following sense: if a given voluminous pair $a, b$ concatenate to make some $c$, then any pair which instantiates the same volumes as $a$ and $b$, respectively, must concatenate to make something with the same volume as $c$. That is,

**Lemma 13.** *If $(a,b) \circ c$, $(d,e) \circ f$, where $a \approx d$ and $b \approx e$, then $c \approx f$.*[55]

*Proof.* Suppose that $(a,b) \circ c$, $(d,e) \circ f$, $a \approx d$, and $b \approx e$. We'll show that $c \approx f$ by reductio:

That is, suppose $c \not\approx f$. By (Totality), either $c \prec f$ or $f \prec c$. Suppose WLOG that $c \prec f$. So $f$ has a part, $c' \approx c$. By (Properly Extensive), there exists some part $x$ of $f$ such that $(c', x) \circ (f)$.

By (Properly Extensive) and $(a,b) \circ (c)$, $c'$ has a part $a' \approx a$. By (Properly Extensive), there exists some part $y$ of $c'$ such that $(a', y) \circ (c')$. By (Additivity) and $a' \approx a$, $y \approx b$.

Since $(a', y) \circ c'$ and $(c', x) \circ f$, it follows that $(a', [x \oplus y]) \circ f$, where $[x \oplus y]$ is the fusion of $x$ and $y$. $a' \approx d$ (since $a \approx d$), so, by $(d,e) \circ f$ and (Additivity), $[x \oplus y] \approx e$.

However, $(y,x) \circ [x \oplus y]$ (since $x$ and $y$ are voluminous and don't overlap), which, by (Within-Object Archimedean), means $[x \oplus y] \not\approx y$. But $y \approx b$ (from above), and $b \approx e$! So, by ($\approx$ Trans), $[x \oplus y] \approx e$ and $[x \oplus y] \not\approx e$. Contradiction!

Therefore $c \approx f$. Hence, if $(a,b) \circ c$, $(d,e) \circ f$, $a \approx d$, and $b \approx e$, then $c \approx f$. □

The '$\preceq$' relation is, by definition, reflexive (since both the '$\approx$' relation and the parthood relation are reflexive). I said, on page 18, that (Properly Extensive) establishes the transitivity

[55]The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

of $\prec$. The transitivity of $\preceq$ follows from ($\approx$ Trans) plus the transitivity of $\prec$. Lemma 14, below, is the last step in establishing that the '$\preceq$' relation induces a total order on the set of equivalence classes determined by $\approx$. That is, it shows that the "$X$ has an element which bears $\preceq$ to an element of $Y$" is an antisymmetric relation between sets of same-volumed objects. Put more simply, it says that:

**Lemma 14.** *If $b \preceq a$ and $a \preceq b$, then $a \approx b$.*[56]

*Proof.* If $x \prec y$ then $x \preceq y$. Hence, it suffices to show that '$\prec$' is *a*symmetric. That is, that $b \prec a$, then $\neg(a \prec b)$—or, equivalently, if $b \prec a$, there are no parts of $b$ which bear $\approx$ to $a$.

Suppose otherwise, for reductio. That is, suppose $b \prec a$ but $b$ has a part, $a' \approx a$. By (Properly Extensive), since $a'$ is a voluminous part of $b$, there must exist some other voluminous part $c \approx c$ such that $a' \circ c = b$. However, by (Properly Extensive) and $b \prec a$, $a$ must have a part $b' \approx b$ and another part, $d \approx d$, such that $b' \circ d = a$.

But, by (Properly Extensive) and (Additivity), $b'$ must partition into non-overlapping parts $a'' \approx a$ and $c' \approx c$ such that $a'' \circ c' = b'$. Since $c'$ and $d$ (both parts of $a$) are voluminous and don't overlap, it follows, by (V-Comb), that their mereological fusion, $(c' \oplus d)$, is voluminous. However, this means that $a$ is the fusion of two non-overlapping voluminous parts, $a''$ and $(c' \oplus d)$, i.e. $a'' \circ (c' \oplus d) = a$. But the Within-Object Archimedean Assumption implies that there are no zero magnitudes of volume (see page 18 in the main text), i.e. that there can be no $xy \circ z$ such that $y \approx y$ and $x \approx z$. But, this means that, given $a'' \circ (c' \oplus d) = a$, it must be that $a \not\approx a''$. Contradiction! So $b$ does not have a voluminous part $\approx a$. □

We'll call a copy of $x$ something with the same volume as $x$. What Lemma 15 says is that the fusion of $n$-many non-overlapping copies of $x$ has the same volume as the fusion of $m$-many non-overlapping copies of $x$ just in case $n = m$. If we wanted to sound like Euclid, we'd say something like this: According to Lemma 15, equal numbers of equal volumes are equal.

**Lemma 15.** *Let $S$ be a set of $n$ objects, all of which bear $\approx$ to some $a$, none of which overlap. Let $S^*$ be another such set with the same cardinality. $fus(S) \approx fus(S^*)$ (where '$fus(X)$' denotes the fusion of all the members of the set, $X$).*[57]

*Proof.* If $n = 1$, then $S$ and $S^*$ are each singletons whose sole member is $\approx a$. $fus(S) \approx a$ and $fus(S^*) \approx a$. By ($\approx$ Trans), $fus(S) \approx fus(S^*)$.

Now we show that, if $fus(S) \approx fus(S^*)$ for some $n \in \mathbb{Z}$, then $fus(S) \approx fus(S^*)$ for $n+1$.

Let $S = \{x_1, x_2, ..., x_{n+1}\}$ and $S^* = \{y_1, y_2, ..., y_{n+1}\}$. Since $S \setminus \{x_{n+1}\}$ and $S^* \setminus \{y_{n+1}\}$ are sets of $n$-many non-overlapping objects all $\approx a$, by our assumption: $fus(S \setminus \{x_{n+1}\}) \approx fus(S^* \setminus \{y_{n+1}\})$.

[56] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[57] The proof of this lemma directly depends on Lemma 13. For the full map of Lemma interdependence, see figure 4, p. 59.

Since $x_{n+1} \approx a$ and $y_{n+1} \approx a$, $x_{n+1} \approx y_{n+1}$ by ($\approx$ Trans). Note that $fus(S)$ is just the fusion of $fus(S \setminus \{x_{n+1}\})$ and $x_{n+1}$, i.e. $(fus(S \setminus \{x_{n+1}\}), x_{n+1}) \circ (fus(S))$. Similarly $(fus(S^* \setminus \{y_{n+1}\}), y_{n+1}) \circ (fus(S^*))$. By Lemma 13, since $fus(S \setminus \{x_{n+1}\}) \approx fus(S^* \setminus \{y_{n+1}\})$ and $x_{n+1} \approx y_{n+1}$, it follows that $fus(S) \approx fus(S^*)$.

So if $fus(S) \approx fus(S^*)$ for some $n \in \mathbb{Z}$, then $fus(S) \approx fus(S^*)$ for $n+1$. Hence, since $fus(S) \approx fus(S^*)$ for $n = 1$, it follows that $fus(S) \approx fus(S^*)$ for all cardinalities, $n \in \mathbb{Z}^+$. □

## B.4 Proper Extensiveness

This section proves that the mereological expression of proper extensiveness in 3.6 entails the second-order version of these conditions in my "Properly Extensive Quantities", and vice versa (given some reasonable assumptions).

The expression of proper extensiveness I presented in Perry (2015) was second-order in that it concerned quantitative ordering and summation relations between the determinate magnitudes of a given properly extensive quantity. Here are the necessary conditionals, with notation modified for clarity and to fit with the rest of this chapter, that I took to characterize proper extensiveness:

(Additive 'Less') $$\textsc{Less}(V_m, V_n) \rightarrow \forall x \forall y((V_n(x) \wedge V_m(y)) \rightarrow \neg Pxy)$$

(Additive 'Sum') $$\textsc{Sum}(V_m, V_n, V_r) \rightarrow \forall x \forall y \forall z((V_m(x) \wedge xy \circ z) \rightarrow (V_r(z) \leftrightarrow V_n(y)))$$

(Extensive 'Less') $$\textsc{Less}(V_m, V_n) \rightarrow \forall x(V_n(x) \rightarrow \exists y(y \neq x \wedge V_m(y) \wedge Pyx))$$

(Extensive 'Sum') $$\textsc{Sum}(V_m, V_n, V_r) \rightarrow \forall x(V_r(x) \leftrightarrow \exists y \exists z(V_m(y) \wedge V_n(z) \wedge yz \circ x))$$

There's a debate to be had about exactly how to define the second-order relations, 'Less' and 'Sum', between determinate volume properties ($V_i$'s) in terms of the first order relations between voluminous objects. It's likely that the right answer would involve some modal notions (e.g., saying that, necessarily, a given property is instantiated by some object $o$ if and only if $o$ has parts with such-and-such other volume properties). I won't weigh in on this here.

Rather, I'll rely on the assumption of some simple bridge laws, saying that the first-order summation and ordering relations should obtain between objects just in case the corresponding second-order relations hold between their respective volume. That is, I will assume the biconditionals "$c$ is as voluminous as $a$ and $b$ put together iff their respective volumes stand in the $\textsc{Sum}(x, y, z)$ relation" and "$a$ is less voluminous than $b$ just in case $a$'s volume bears $\textsc{Less}(x, y)$ to $b$'s volume". It's a necessary condition of any adequate second-order theory of quantity, regardless of how or whether it defines the second-order quantitative relations, that it satisfy (for all volume magnitudes $V_m, V_n, V_r$), (Bridge 1) $\forall x, y((V_m(x) \wedge V_n(y)) \rightarrow (x \prec y \leftrightarrow \textsc{Less}(V_m, V_n)))$

and

(Bridge 2)
$\forall x, y, z((V_m(x) \wedge V_n(y) \wedge V_r(z)) \rightarrow (z \text{ is as voluminous as } x \text{ and } y \text{ put together} \leftrightarrow \textsc{Sum}(V_m, V_n, V_r)))$

While the M-R account is technically a property-theoretic account, its only appeals to properties concern whether or not $a$ and $b$ instantiate the *same* volume magnitude ($a \approx b$). As such, it is ill-equipped to deal with cases of uninstantiated volume properties. In what follows, I will assume that any volume magnitude which appears in a logically atomic sentence (e.g. instances of '$\textsc{Less}(x, y)$' or '$\textsc{Sum}(x, y, z)$') is instantiated by at least one object.[58]

Given these assumptions and our two bridge laws, I will show (Lemmas 16–19) that the conditionals I introduce in Perry (2015) follow from the M-R account of volume, specifically (Additivity) and (Properly Extensive). I'll then show that (Additivity) and (Properly Extensive) can be derived from those four conditionals along with some reasonable assumptions (Lemmas 20–21).[59]

**Lemma 16.** , **(Additive 'Less')** *If* $\textsc{Less}(V_m, V_n)$ *then* $\forall x \forall y((V_n(x) \wedge V_m(y)) \rightarrow \neg Pxy)$

*Proof.* Suppose that $\textsc{Less}(V_a, V_b)$. Consider an arbitrary pair $a$ and $b$ such that $V_a(a)$ and $V_b(b)$. By (Bridge 1), this implies that $a \prec b$.

By Lemma 14, if $a \prec b$, there are no parts of $a$ which bear $\approx$ to $b$. Since $b \approx b$, this implies that $\neg P(ba)$. So $\neg P(ba)$ follows from the assumption that $V_a(a)$ and $V_b(b)$. Since this was shown for an arbitrary pair, $a$ and $b$, it's true for all such pairs, i.e. $\forall x \forall y((V_b(x) \wedge V_a(y)) \rightarrow \neg Pxy)$.

□

**Lemma 17.** , **(Additive 'Sum')** *If* $\textsc{Sum}(V_m, V_n, V_r)$ *then* $\forall x \forall y \forall z(V_m(x) \wedge xy \circ z \rightarrow (V_r(z) \leftrightarrow V_n(y)))$

*Proof.* Suppose that $\textsc{Sum}(V_a, V_b, V_c)$. By the instantiation assumption, $V_a(a)$, $V_b(b)$, and $V_c(c)$ for some $a$, $b$, and $c$. By (Bridge 2), $c$ is as voluminous as $a$ and $b$ put together. Or, equivalently, there exist some $a' \approx a$ and $b' \approx b$ such that $a'b' \circ c$.

Now consider an arbitrary trio: $s$, $t$, $u$, such that $V_a(s)$ (i.e. $s \approx a$) and $st \circ u$. It suffices to show that $V_c(u) \leftrightarrow V_b(t)$ (equivalently, $u \approx c \leftrightarrow t \approx b$).

[58] A discussion which tackles the issues with uninstantiated magnitudes head-on would require a detailed account of the modal profile of these properties, specifically under what conditions it's necessary that they be instantiated or not instantiated, and so on. I've already said that getting into that debate would take us too far afield.

[59] The proofs for lemmas 16 and 18 directly depend on Lemma 14, while the proofs for lemmas 17 and 19 depend on Lemma 13. For the full map of Lemma interdependence, see figure 4, p. 59.

*Left to right:* Suppose that $V_b(t)$, i.e. that $t \approx b$. Since $s \approx a$, $t \approx b$, $st \circ u$ and $ab \circ c$, it follows from Lemma 13 (Unique Sum Lemma), that $u \approx c$. Since $V_c(c)$, this implies that $V_c(u)$.

*Right to left:* Suppose that $V_c(u)$, i.e. that $u \approx c$. By (Additivity) and $ab \circ c$: if $u \approx c$ and $st \circ u$, then $s \approx a$ just in case $t \approx b$. Since we've supposed that $V_a(s)$, i.e. that $s \approx a$, it follows that $t \approx b$. Since $V_b(b)$, this implies that $V_b(t)$.

So $V_c(u) \leftrightarrow V_b(t)$ follows from the assumption that $V_a(s)$ and $st \circ u$. Since this was shown for an arbitrary trio $s$, $t$, and $u$, this conditional holds universally. □

**Lemma 18.** , **(Extensive 'Less')** *If* $\textsc{Less}(V_m, V_n)$ *then* $\forall x(V_n(x) \rightarrow \exists y(y \neq x \wedge V_m(y) \wedge Pyx))$

*Proof.* Suppose that $\textsc{Less}(V_a, V_b)$. Consider an arbitrary $b$ such that $V_b(b)$. By the instantiation assumption, there exists some $a$ such that $V_a(a)$. It suffices to show that there exists some part, $c$, of $b$ such that $c \approx a$ and $c \neq b$.

By (Bridge 1), $a \prec b$. By the definition of $\prec$, there exists a part, $c$, of $b$ such that $c \approx a$. Since $V_a(c)$ and $V_a \neq V_b$, $c \not\approx b$ so $c \prec b$. By Lemma 14, if $c \prec b$, then there are no parts of $c$ which bear $\approx$ to $b$. If $c$ were identical to $b$, then $c$ would have a part that bears $\approx$ to $b$, hence $c \neq b$. Since this was shown for an arbitrary $b$ such that $V_b(b)$, it's true in general, i.e. $\forall x(V_b(x) \rightarrow \exists y(y \neq x \wedge V_a(y) \wedge Pyx))$. □

**Lemma 19.** , **(Extensive 'Sum')** *If* $\textsc{Sum}(V_m, V_n, V_r)$ *then* $\forall x(V_r(x) \leftrightarrow \exists y \exists z(V_m(y) \wedge V_n(z) \wedge yz \circ x))$

*Proof.* Suppose that $\textsc{Sum}(V_a, V_b, V_c)$.

*Left to Right:* Pick some arbitrary object $c$ such that $V_c(c)$. By our original supposition plus the instantiation assumption, there exists some $a$ and $b$ such that $V_a(a)$ and $V_b(b)$. By (Bridge 2), $c$ is as voluminous as $a$ and $b$ put together. Or, equivalently, there exist some $a' \approx a$ and $b' \approx b$ such that $a'b' \circ c$. So the left to right half of the biconditional is established.

*Right to left:* Let $d$ be an arbitrary entity such that there exists some $x$ and $y$ such that $V_a(x)$, $V_b(y)$, and $xy \circ d$. By our original supposition plus the instantiation assumption, there exists some $c$ such that $V_c(c)$. By (Bridge 2), $c$ is as voluminous as $x$ and $y$ put together, i.e. there exists some $x' \approx x$ and $y' \approx y$ such that $xy \circ c$. By Lemma 13 (Unique Sum) if $xy \circ d$ and $x'y' \circ c$, where $x' \approx x$ and $y' \approx y$, it follows that $d \approx c$. Since $V_c(c)$, this means that $V_c(d)$. So the right to left half of the biconditional is established.

Since both directions of the biconditional are established given an arbitrary individual satisfying each antecedent, the biconditional holds universally. That is, $\forall x(V_c(x) \leftrightarrow \exists y \exists z(V_a(y) \wedge V_b(z) \wedge yz \circ x))$. □

The next step will be to prove that the axioms which characterize volume's proper extensiveness on the M-R account follow from my original formulation, given the bridge laws and some reasonable assumptions (which I'll go into during the proof of Lemma 21, below). I do not presuppose the axioms of the M-R account of volume, with the exception of the volume combination principle, (refvcomb), which says that if a pair of voluminous entities are put together in the right way, their fusion is voluminous.

**Lemma 20.** *The axiom, (Additivity)*

(Additivity) $$a \approx a \wedge b \approx b \wedge ab \circ c \;\rightarrow \\ \forall x \forall y \forall z\big(x \approx c \wedge yz \circ x \rightarrow \;(y \approx a \rightarrow z \approx b)\big)$$

*follows from* (Additive 'Less'), (Additive 'Sum'), (Extensive 'Less'), *and* (Extensive 'Sum').[60]

*Proof.* Suppose there exist some $V_a, V_b$ such that $V_a(a)$ and $V_b(b)$ and $ab \circ c$. By (V-Comb) $\exists V_c$ such that $V_c(c)$. Therefore, by definition, $c$ is as voluminous as $a$ and $b$ put together. By (Bridge 2) $\textsc{Sum}(V_a, V_b, V_c)$.

We want to show that, for any $x, y, z$ such that $V_c(x)$ and $yz \circ x$, $V_a(y) \rightarrow V_b(z)$. To show this, suppose $x$ is such that $V_c(x)$ and $yz \circ x$. Further, suppose that $y \approx a$ (i.e. $V_a(y)$). Suffices to show that $z \approx b$.

Since $\textsc{Sum}(V_a, V_b, V_c)$, by (Additive 'Sum') $yz \circ x$ and $V_a(y)$ imply that $V_c(x) \leftrightarrow V_b(z)$. $V_c(c)$, and we've assumed that $V_c(x)$, so $x \approx c$. So it it follows that $V_b(z)$ and, since $V_b(b)$, $z \approx b$. □

**Lemma 21.** *The axiom, (Properly Extensive)*

$$a \approx a \wedge Pab \wedge b \approx d \rightarrow (a \approx b \vee \exists x \exists y(x \approx a \wedge y \approx y \wedge xy \circ d))$$

*follows from* (Additive 'Less'), (Additive 'Sum'), (Extensive 'Less'), *and* (Extensive 'Sum'), *plus the assumption that* $\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r))$ *for all* $V_m$, $V_n$, *and* $V_r$.[61]

*Proof.* Let $a$, $b$, and $d$ be arbitrary individuals such that $a$ is a part of $b$ and there exist some volume properties, $V_a$, $V_b$, such that $V_a(a)$, $V_b(b)$, and $V_b(d)$.

We want to show that either $a \approx b$ or $\exists x \exists y \exists V_y(x \approx a \wedge V_y(y) \wedge xy \circ d)$. Either $V_a = V_b$ or not. We reason by cases: If $V_a = V_b$, then $a \approx b$ by the definition of $\approx$. Suppose, instead, that $V_a \neq V_b$.

If $V_a \neq V_b$, then $a \not\approx b$. By the definition of '$\prec$', since $Pab$ and $a \not\approx b$, $a \prec b$. By (Bridge 1), this means that $\textsc{Less}(V_a, V_b)$. Since $V_b(d)$, (Bridge 1) also implies that $a \prec d$. By the definition of '$\prec$', there's a part, $a'$, of $d$ such that $a' \approx a$.

[60] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[61] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

This establishes that $d$ has a proper part $a' \approx a$. However, we need an additional assumption to get to (Properly Extensive). Specifically, we need to assume that $\forall V_m, V_n(\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r)))$. This makes sense. The conditionals presented in Perry (2015) were never meant to stand alone. Rather, they serve to connect a Mundy-style second-order account of quantitative structure to the first-order mereological structure. On a Mundy-style account of volume, primitive axioms govern the distribution of the second-order summation and ordering relations over the first-order volume properties. The analogue of "$\forall V_m, V_n(\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r)))$" for mass is entailed by the account in Mundy (1987), so it's reasonable to assume it would hold here.

If we accept this conditional, then we can infer, from $V_a(a)$, $V_b(d)$, and $\textsc{Less}(V_a, V_b)$, that there exists some volume property, $V_x$, such that $\textsc{Sum}(V_a, V_x, V_b)$. The rest follows from (Extensive '$\textsc{Sum}$'). Specifically, $\textsc{Sum}(V_a, V_x, V_b)$ and $V_b(d)$ imply that $\exists y \exists z(V_a(y) \wedge V_x(z) \wedge yz \circ d)$. Hence $\exists y \exists z(y \approx a \wedge z \approx z \wedge yz \circ d)$.

Since it follows from both $V_a = V_b$ and its negation, the disjunction, $a \approx b$ or $\exists x \exists y \exists V_y(x \approx a \wedge V_y(y) \wedge xy \circ d)$, follows from the supposition that $a$ is a voluminous part of $b$ and $b \approx d$. Since the conditional holds for an arbitrary $a$, $b$, and $d$, it holds for all such trios. □

## B.5 Continued Fractions

In this section I'll show that every real number can be uniquely expressed as a simple continued fraction. These are not new results, but in case the reader is not familiar with these features of continued fractions, there are certain easy and intuitive ways to make it clear that they have these properties.

Let $n \in \mathbb{R}$ be some real number. First I will show that $n$ can be expressed as a simple continued fraction, then I will show that this expression is unique. This presentation closely follows Chrystal (1886).

Consider the integer $a_0$, where $a_0$ is the greatest integer such that $a_0 \leq n$. So

$$n = a_0 + \frac{1}{n_1} \tag{25}$$

where $n_1 > 1$ is some real number. Now let $a_1$ be the greatest integer $\leq n_1$. Then

$$n_1 = a_1 + \frac{1}{n_2} \tag{26}$$

where $n_2 > 1$ as before. Again, let $a_2$ be the greatest integer $\leq n_2$, then

$$n_2 = a_2 + \frac{1}{n_3} \tag{27}$$

This process will terminate if some $n_i$ is an integer, for then we'd have

$$n_i = a_i \tag{28}$$

and there would be no $n_{i+1}$. Also, since $n_i$ is the result of a process like the one above, it must be $> 1$, and so $a_i > 1$. If there is no such $n_i$ the process will not terminate.

From these we get that

$$n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{29}$$

or

$$n = a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3+} \cdots \tag{30}$$

Regardless of whether this process terminates, we have resulted in either a terminating or infinite simple continued fraction expression of $n$, since $n \in \mathbb{R}$ is an arbitrary real number, we can conclude that all real numbers can be expressed as simple continued fractions.

Now to prove that the continued fraction expression of a real number is unique. It suffices to show that for any $n$ such that

$$n = a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3+} \cdots = a_0' + \frac{1}{a_1'+} \frac{1}{a_2'+} \frac{1}{a_3'+} \cdots \tag{31}$$

Each term is equal, i.e. $a_0 = a_0'$, $a_1 = a_1'$, $a_2 = a_2'$, etc.

We know, from the process by which continued fractions are constructed, that $a_0$ and $a_0'$ are positive integers, and that $\frac{1}{a_1+} \frac{1}{a_2+} \cdots$ and $\frac{1}{a_1'+} \frac{1}{a_2'+} \cdots$ are both positive (or zero) and $< 1$ (since every $a_i$ and $a_i'$ is positive). Suppose (WLOG) that $a_0 < a_0'$. Then $a_0' \geq a_0 + 1$ (since they are both integers).

But then

$$a_0' + \frac{1}{a_1'+} \frac{1}{a_2'+} \frac{1}{a_3'+} \cdots \geq (a_0 + 1) + \frac{1}{a_1'+} \frac{1}{a_2'+} \frac{1}{a_3'+} \cdots \tag{32}$$

Since, by (31) the primed and unprimed continued fractions are equal, it follows that

$$a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3+} \cdots \geq a_0 + 1 + \frac{1}{a_1'+} \frac{1}{a_2'+} \frac{1}{a_3'+} \cdots \tag{33}$$

Subtracting out $a_0$, we have

$$\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots \geq 1+\frac{1}{a'_1+}\frac{1}{a'_2+}\frac{1}{a'_3+}\cdots \tag{34}$$

But we know that $\frac{1}{a'_1+}\frac{1}{a'_2+}\cdots \geq 0$, hence $\frac{1}{a_1+}\frac{1}{a_2+}\cdots \geq 1$. But we know that $\frac{1}{a_1+}\frac{1}{a_2+}\cdots < 1$. So the assumption that $a_0 < a'_0$ leads to a contradiction, hence $a_0 \geq a'_0$. Parallel reasoning shows that $a'_0 \geq a_0$. Hence $a_0 = a'_0$.

If the first terms are equal, then

$$\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots = \frac{1}{a'_1+}\frac{1}{a'_2+}\frac{1}{a'_3+}\cdots \tag{35}$$

which means that their inverses[62] are equal, i.e.

$$a_1+\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots = a'_1+\frac{1}{a'_2+}\frac{1}{a'_3+}\frac{1}{a'_4+}\cdots \tag{36}$$

Here $a_1$ and $a'_1$ are both positive integers, and $\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\ldots$ and $\frac{1}{a'_2+}\frac{1}{a'_3+}\frac{1}{a'_4+}\ldots$ are positive (or zero) and $< 1$. By the same reasoning as before, $a_1 = a'_1$

We can continue this process, and can show that each $a_i = a'_i$, even if the fractions never terminate.[63]

Hence continued fractions can be used to uniquely pick out real numbers. Continued fractions are structured such that they can be expressed as lists of positive integers (i.e. $\langle a_0, a_1, a_2, \ldots \rangle$) which may or may not terminate (and where the final integer if there is one, is $> 1$). The ratio procedure for a pair of voluminous entities, generates a list of positive integers, which may or may not terminate (and where the final integer, if there is one, is $> 1$). Hence, the list of integers generated by the ratio procedure for any voluminous pair *uniquely* corresponds to the continued fraction expansion of a positive real number.

[62] What does it mean to take the inverse of a continued fraction, given that these are things which may be infinitely long? The operation is actually not as mysterious as it might seem. We are taking the inverse of a fraction whose numerator is = 1. In this case, the inverse is just the denominator of that fraction.

[63] Technically, one would have to prove that infinite simple continued fractions converge in the right way, so that the reasoning, above, using the continued fractions make sense even when they don't terminate. This is a well known result, and can be seen in (Chrystal, 1886, p. 441) , and (Davenport, 1962, p. 93).

## C Map of Dependencies Between Lemmas

Here I include a graphical map of the dependency relations between the 21 Lemmas featured in the text.

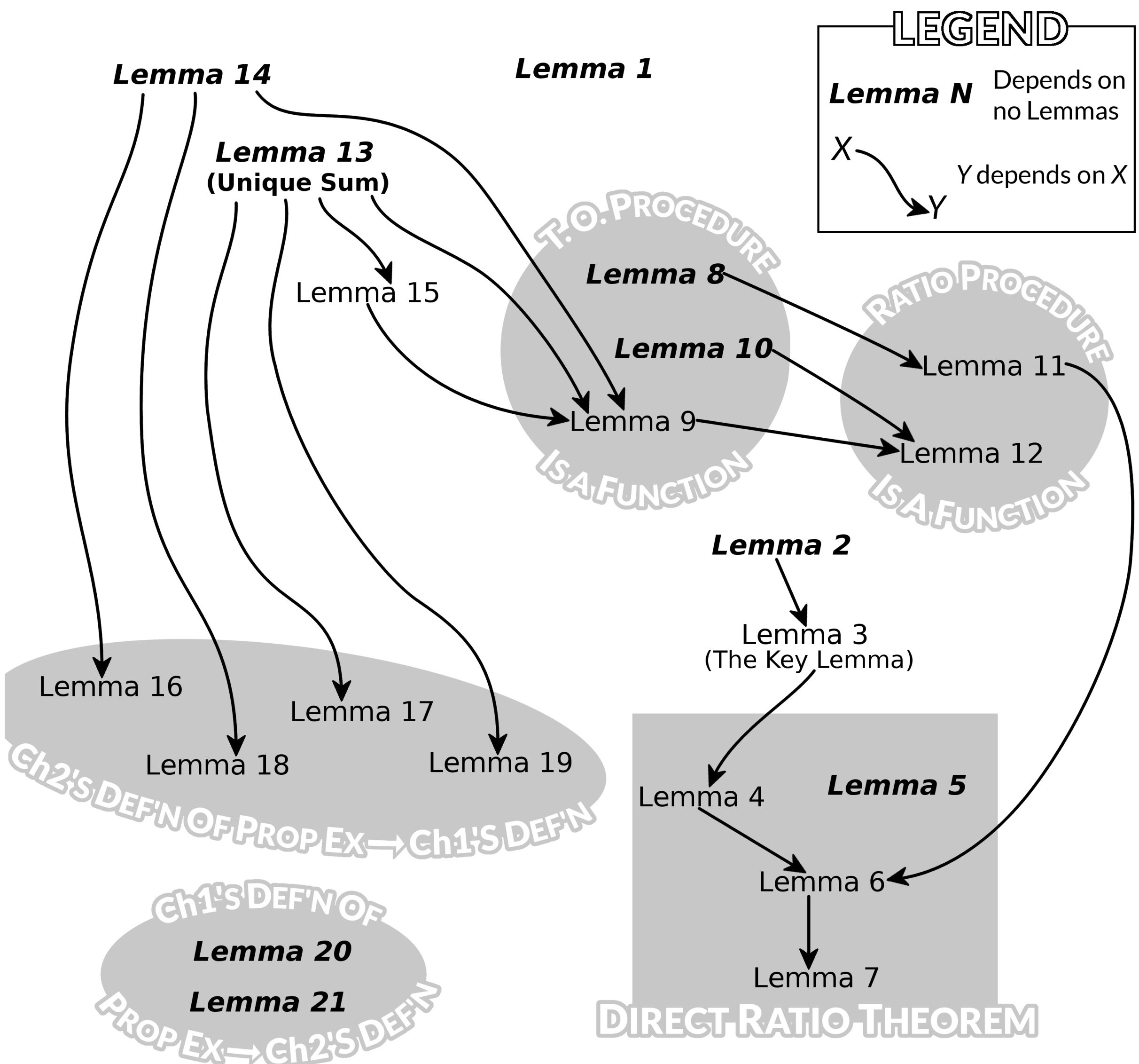


Figure 4: A graphical depiction of the dependency relations between the lemmas proved in the text. Since every dependency arrow follows a downward trajectory, this figure also serves as a visual proof that the dependence of some lemmas on others does not commit us to any circularity.

Lemma 1: p. 20
Lemma 2: p. 32
Lemma 3: p. 33
Lemma 4–7: p. 39–42
Lemma 8–10: p. 44–46
Lemmas 11, 12: p. 48
Lemma 13–15: p. 51
Lemmas 16–19: p. 53–54
Lemma 20, 21: p. 55

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