Mereology and Metricality*

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We represent physical quantities, in science and our everyday practice, using mathematical entities like numbers and vectors. We use a real number and unit to refer to determinate magnitudes of mass or length (like $2kg$, $7.5m$ etc.), and then appeal to the arithmetical relations between those numbers to explain certain physical facts. I cannot reach the iced coffee on the table because the shortest path between it and me is $3ft$ long, while my arm is only $2.2ft$ long, and $2.2 < 3$. The scale at the farmer's market does not tilt because one pan holds a $90g$ tomato while the other holds two strawberries, of $38g$ and $52g$ respectively, and $38 + 52 = 90$. The amount of water that spills out of the tub when Archimedes gets in is 3.5-times greater than what spills out when Archimedes Jr. gets in, because their bodies' volumes are 83.3 and 23.8 cubic decimeters, respectively, and the ratio between 83.3 and 23.8 is $3.5 : 1$ (i.e. $83.3 = 3.5 * 23.8$).

It seems right to say that, while they provide a convenient way to *express* these explanations, the mathematical '$<$' relation, or the '$+$' and '$*$' operations on the real numbers are not *really* part of the physical explanations of these events.[1] They just represent explanatorily relevant features inherent in the physical systems described—i.e. the features of the tomatoes, strawberries, bathtubs, and ancient Greeks involved. A theory of "quantitative structure" is an account of these features, the physical properties and relations *really* doing the explaining.

People have thought[2] the proper account of quantities requires that we give up on the idea that quantitative structure be intrinsic in this way. They've thought that, to the extent predicates like "2π-times as long as" or "three-and-a-half times the volume of" pick out physical relations *at all*, they only be defined in terms of global structural characteristics of the domains of lengthy or voluminous entities,[3] not in terms of how their relata are *in themselves*.

I will show that this is a mistake. This paper defends a theory of quantitative structure that does justice to the intuition that the physical relations which constitute quantitative structure

*This is a condensed version of the second chapter of my dissertation. It has been revised to stand alone. This text, at various points, refers the reader to a formal appendix, which has been omitted from this file. For those interested, a PDF version of this appendix can be found at: zrperry.com/appendix .

[1] This is not *entirely* uncontroversial. Some have tried to defend more sophisticated versions of the claim that mathematical objects directly explain physical facts involving quantities, most recently Knowles (2015).

[2] Most notably Hölder (1901), Krantz et al. (1971), and Arntzenius and Dorr (2012), as well as Mundy (1987) and Eddon (2013). I will also argue that, despite initial appearances to the contrary, this is a commitment of Field (1980) and (1984) for most quantities.

[3] Or, in the case of second-order theories of quantity like Mundy (1987), the total domain of determinate length or volume *properties*.

are intrinsic. I argue that, for *some* quantities—namely, the members of a special class of quantities I call "*properly extensive*"—the explanation for why our mathematical representations are faithful comes from their connection to parthood. Let me give an example of what I mean; consider the following two judgments:

(1) x is shorter than y

(2) x is as long as a part of y

(1) is an instance of an ordering relation on lengthy objects, where the ordering relation is part of what constitutes length's quantitative structure. (2), alternatively expressed as "some part of y has the same length as x", can be broken down into, on the one hand, a *mereological* relation – parthood – and, on the other, the relation denoted by a predicate like "as long as" or "same length as".[4]

In this paper, I will argue that claims like (1) reduce to claims like (2).

1 Quantitative Structure is Parthood Structure

More precisely, I defend the *Mereological-Reductive* (or "M-R") *account of quantitative structure*, which defines (1) as "(2) *and* x and y do *not* have the same length", and gives a definition—in terms of parthood and the sharing of determinate length properties—for *all* the relations which constitute length's quantitative structure.

Many other accounts of quantitative structure introduce a quantity's ordering relation, like (1), or *summation* relation (like what's appealed to in the balance scale example, or discussed, below, in the case of length) as *primitive posits*.[5] Accounts like these, if they want to capture the intuitive connection between (1) and (2), have to posit bridge laws between their primitive relations and the mereological ones. The M-R account avoids this by taking the connection to be definitional. There is a tradeoff, of course. Primitive posits are as adaptable as their axioms allow them to be, and it's easy to generalize an account that makes use of, e.g., primitive ordering relations to apply to any quantity that's ordered. In contrast, the M-R account's definitions of ordering, summation, and metrical ratio relations can only be satisfied by quantities which put the right necessary constraints on the parthood structure of their instances.

In this section, I give an overview of the M-R account of quantitative structure, and argue

[4]On the reading I'm interested in here, "x has the same length as y" means simply that x and y instantiate the *same* (i.e. numerically identical) length property. There are other readings, on which "same length as" is a fundamentally two-place relation that constitutes another part of length's quantitative structure. I discuss this alternative in section 3.2.

[5]E.g. Mundy, Eddon, Bigelow and Pargetter, and (arguably) Russell posit primitive second-order relations, while accounts based on Hölder or Krantz et al. posit primitive ordering relations between, and concatenation operations on, physical objects. (Mundy, 1987), (Eddon, 2013), (Bigelow et al., 1988), (Krantz et al., 1971), (Hölder, 1901), (Russell, 1903).

that the commonly accepted way quantitative and mereological structure can be related, what is sometimes called "extensiveness" or "additivity", is too weak to support this account. The M-R definitions, I argue, apply only to the *properly* extensive quantities, a special sub-class of the extensive quantities which put additional constraints on the possible mereological structure of their instances.

In sections 3 and 4, I present a formal M-R account of the quantity, volume, which takes its proper extensiveness as fundamental and defines the ordering, summation, and metrical relations which constitute its quantitative structure in terms of this connection to mereology. The system also serves as a general schema for M-R accounts of other properly extensive quantities, like length, area, temporal duration, etc. In section 2, I argue that no other theory of quantitative structure does justice to the intrinsicality intuition as it applies to the properly extensive quantities, and present a number of other advantages of the account.

1.1 Summation Structure

If the M-R account is going to be able to do all I've promised it can, it needs to give definitions of the ordering, summation, and metrical ratio relations that captures the idea that they reflect something intrinsic to their relata. This is easy in the case of length ordering, since (2) is a natural reading of (1) and is also an intrinsic relation.

It's less obvious how summation or metrical relations should be defined on this account. A common expression of length summation relations involves talking about length *properties* rather than lengthy objects. We say "x's length is the *sum* of y's and z's lengths". The natural expression of the relation between lengthy *objects* doesn't use terms like 'sum' at all. Rather, it says

(3) x is as long as y and z put together.

(3) has, if anything, more of a mereological ring to it than (1). Indeed, on a literal reading of 'put together', we can gloss (3) as: "x has the same length as an object, o, composed out of y and z put together, would". However, while this reading is a *mereological* relation, it will not do as an analysis of (3). This is because it requires appeal to o, and in particular o's length. But o might not be lengthy; that is, y and z might not be put together in the right way[6] for o to have length (if, e.g., they make a 'T' shape). Or o might be lengthy but not have the *right* length (if y and z have some lengthy overlap, o's length will not be the "sum" of their lengths).

[6]Two lengthy objects are "put together in the right way", intuitively, when they are laid end-to-end. For other quantities, being put together in the right way will amount to something different. Volume, for instance, is simpler than length in this regard; two voluminous entities are "put together in the right way" just in case they don't overlap (or have a "volume-less overlap", where this means either their overlap instantiates $0m^3$, or it's not voluminous at all). I discuss this further in section 3.3.

The M-R account defines summation structure in a different way. It analyzes (3) in terms of y and z's relations to x's parts:

(4) x is composed of a segment as long as y and a segment as long as z, *put together in the right way*.

Here, the M-R account's analysis goes beyond the intuitive mereological upshot of the summation relation. The M-R account understands both the ordering and the summation relations as specifying (among other things) something about the physical makeup of one of their relata. To say that b is shorter than a, or to say that a is as long as b and c put together, is to say something about *a's internal structure*—specifically, whether a has any parts, what the configuration of those parts is relative to each other, and whether they share length properties with b or c. These relations, so defined, are intrinsic to the system composed by their relata. Indeed, they satisfy a stronger condition: since they depend only on the intrinsic properties of *each* of their relata—i.e. on how each relatum is *in itself*—they are not just intrinsic but *internal* relations.

1.2 Constructing Metric Structure

The M-R account, similarly, defines ratio relations like "twice as long as" or "4.6-times the volume of" in terms of mereological relations and the sharing of intrinsic properties. Though our expressions of them appeal to numerical ratios like 2 : 1 or 4.6 : 1, the physical ratio relations should be understood as relations between concrete physical objects, *not* as relating objects to numbers. If that's right, then there are an infinitude of distinct, two-place, ratio relations; i.e. "2-times the volume of" and "4.6-times the volume of" are distinct relations between voluminous objects. The M-R account gives a reductive definition, in terms of mereology and the sharing of intrinsic volume properties, for *each* such relation, by way of a procedure which pairs ratio relations with their mereological analyses.

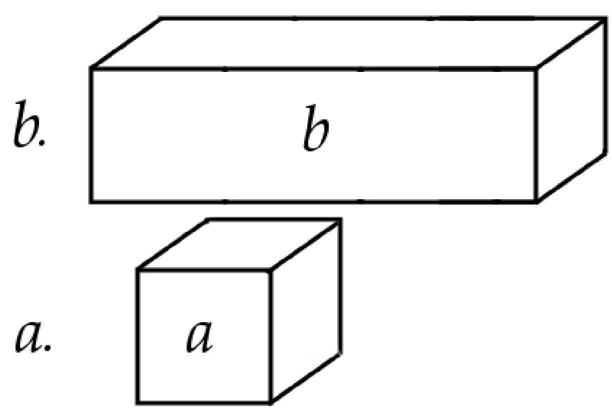

Figure 1: Two voluminous entities.

The "ratio procedure", performed on an ordered pair of voluminous objects, specifies the M-R account's definition of the ratio relation they stand in. Let me give an example of how this works.[7] Suppose we want to determine the volume ratio of b to a (how much more voluminous b is than a). We perform the ratio procedure on a and b.

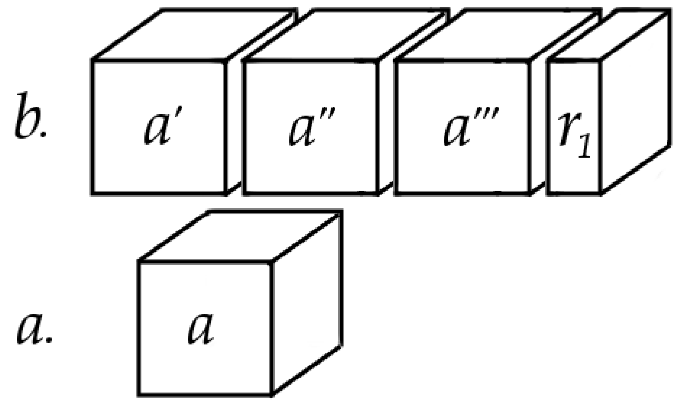

Figure 2: Step 1, Taking a out of b.

First, we "take a out of b", where this just means that we partition b into as many non-overlapping copies of a (i.e. parts with the same volume as a) as we can. In this case, that number is 3. Then there is a part of b which is our remainder, r_1, which is smaller than a. For step two, we do the same thing but taking r_1 out of a, which yields 2 non-overlapping copies and another remainder r_2. The third step follows this pattern, taking r_2 out of r_1. r_1 is composed of 2 non-overlapping copies of r_2 with no remainder. Since there's no remainder, we stop.

This procedure determines the complex mereological property which will be the M-R account's analysis of the "volume ratio" of b to a (let's say that b "partitions into" some class of its parts iff no two of the members of that class overlap and b is their fusion):

> $\exists x_1, x_2$(b partitions into: 3 parts with the same volume as a, and another part, x_1) $\wedge$ (a partitions into: 2 parts with the same volume as x_1, and another part, x_2) $\wedge$ (x_1 partitions into: 2 parts with the same volume as x_2).

How does this give us the volume ratio between a and b? Taking a out of b tells us approximately how much bigger b is than a. Taking r_1 out of a tells us approximately how much bigger a is than r_1. This, in turn, gives us a better approximation of how much bigger b is than a. Each time we repeat this procedure, we get a better and better approximation. If the procedure terminates, we have a perfect approximation. Indeed, r_2 goes evenly into a and b. From the procedure we can deduce that a is composed of 5 non-overlapping copies of r_2, and b 17 copies.

[7] In section 4.2, I formally define this procedure and show how the axioms of my account of properly extensive quantities entail that it always have a well-defined output.

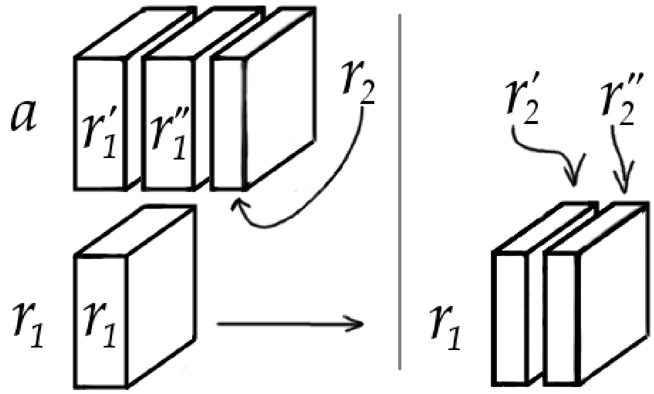

Figure 3: Step 2 and Step 3

So the ratio of b to a is 17/5, i.e. $b = \frac{17}{5} * a = 3.4 * a$. Indeed, "$b$ partitions into 17 parts, all with the same volume as r_2, while a partitions into 5 such parts" amounts to the same thing as the definition given above. Why not just use *that* as the definition for volume ratios, then?

Here's why: The ratio procedure is not guaranteed to terminate, and if it does not terminate, it cannot output a final remainder (like our r_2). However, there's another way to determine the ratio between a and b from this procedure that doesn't require appeal to the final remainder. Recall that the numbers, $\langle 3, 2, 2 \rangle$, output by the procedure, count up certain non-overlapping parts of b, a, and r_1. We can use these to construct what is called a "simple continued fraction":

$$\frac{17}{5} = 3 + \cfrac{1}{2 + \cfrac{1}{2}}$$

The list of integers output by this procedure is what is sometimes called an "*anthyphairetic ratio*".[8] Continued fractions are one way to express this sort of ratio. Even when the ratio procedure does not terminate, it will still output a list of integers that count up the relevant sets of non-overlapping parts of a and b and the various non-final remainders.[9] The only difference is that, when the procedure does not terminate, we get an infinitely long list. This is okay because continued fractions can, in fact, be continued indefinitely, and infinite simple continued fractions *also pick out unique real numbers*! It's this formal feature which allows the mereological relations generated by this procedure to serve as the definitions for volume ratio relations.

[8] The ratio procedure, as defined in section 4.2, is closely related to the process of *anthyphairesis*, a term derived from the Greek for "reciprocal subtraction". Cf. (Fowler, 1987).

[9] The procedure's uniqueness is proved in appendix section B.2, Lemma 12.

1.3 Proper Extensiveness

I mentioned before that the M-R account applies only to quantities that put the right constraints on the possible parthood structure of their instances. Here's what that means: If the M-R definitions of an ordering relation, "LESS-Q", or summation relation, "Q-SUM", are to be any good, *at the very least* the definiens and definiendum must be necessarily coextensive. That is, a quantity, Q, is amenable to the M-R account only if it satisfies:

(5) $\Box$(x is LESS-Q than y $\leftrightarrow$ x and y have different Q-properties, but x has the same Q property as some part of y)

(6) $\Box$(y and z Q-SUM to z $\leftrightarrow$ x can be partitioned into two parts that are *put together in the right way* and which have the same Q-properties as y and z, respectively)

As well as the analogous necessary biconditionals for the ratio relations.

This means that for many (indeed *most*) quantities, the account cannot get off the ground. An M-R account of temperature, for instance, would get the quantitative relations almost entirely wrong. The ice in the freezer, at 30° Fahrenheit, is less warm than $212°F$ water boiling on the stove. But this fact about temperature ordering clearly doesn't mean that the ice in the freezer is as warm as some proper part of the water on the stove!

What about quantities that, unlike temperature, put significant constraints on the mereology of their instances? Additive (also called *extensive*[10]) quantities are ones where, intuitively, wholes inherit their Q-properties from the Q-properties their parts. More precisely, Q is additive just in case: whenever x and y instantiate Q-properties, and are "put together in the right way", the mereological fusion of x and y instantiates the "sum" of their Q-properties. Being additive is necessary for a quantity to admit of an M-R account of its structure (a quantity is additive just in case it satisfies the right-to-left direction of both (5) and (6).), but it is not sufficient.

Here's why: Consider the additive quantity, mass. On the standard model of particle physics, there are fundamental particles with different masses, like the electron (approx. $9.19 \times 10^{-31} kg$), and the muon (approx. $1.88 \times 10^{-28} kg$). On a straightforward interpretation of this theory, Ellen the electron and Miriam the muon are mereological simples. This is inconsistent with both (5) and (6), since Ellen does not have a part with the same mass as Miriam, yet the standard model is not (and should not be) taken to be inconsistent with mass's additivity. So,

[10] The IUPAC (The International Union of Pure and Applied Chemistry) "Green Book" – part of a series of manuals meant to "provide a readable compilation of widely used terms and symbols" and promote "good practice of scientific language" – defines extensiveness as follows: "A quantity that is additive for independent, noninteracting subsystems is called *extensive*". *p*.6. There has been little discussion in the philosophical literature about additivity itself. To the extent it has been discussed by contemporary philosophers, they have followed scientific practice, cf. (Busse, 2009), (Johansson, 1996), and (McQueen, 2015).

while additive quantities have a very close connection to mereology, a quantity's being additive is not sufficient to support an M-R account of its structure.

In Perry (2015), I argue that some quantities put stronger constraints on the mereology of their instances than what additivity requires. These quantities I call "*properly* extensive"[11] (recall that the unmodified term 'extensive' is equivalent to 'additive'). The properly extensive quantities comprise a sub-class of the extensive quantities (quantities which are extensive but not *properly* so I call "merely additive"). Some quantities we classify as additive are, I claim, also properly extensive—specifically length, area, volume, temporal duration, and the invariant relativistic interval. Properly extensive quantities put stronger constraints on the relationship between quantitative structure and mereology than merely additive ones, like charge or mass, do.[12] Most importantly, properly extensive quantities, intuitively, satisfy (5) and (6).

The connection that properly extensive quantities have to the parthood structure of their instances is what makes them amenable to the M-R account's definitions of the quantitative relations. This amounts to more than just a restriction on the range of applicability of the account. It tells us how and why the M-R definitions work when they do. That is, the M-R account, on its own, only tells us *that* our representations of Q are faithful insofar as the structure of the mathematical entities we appeal to mirrors the mereological structure of that quantity's instances, and the distribution of intrinsic Q-properties over that structure. The proper extensiveness of Q tells us *why* there's a necessary correspondence of this sort between the mathematical and the mereological. This is what it means to say that the success of our mathematical representations of these quantities is explained by their connection to parthood.

1.4 The Rest of the Paper

Sections 3 and 4 make good on the promises made in this section. There, I present a formal M-R account of volume which takes the necessary constraints obeyed by properly extensive quantities as axioms.[13] From the assumption of volume's proper extensiveness, and very little

[11]There is *some* reason to suspect that something like what I call "proper extensiveness" is what Meinong (1896) calls "divisible quantities" ('*Teilbare Größen*'). However, there is also evidence that this term was used by Meinong to indicate infinite divisibility rather than a correspondence between quantitative structure and mereological structure. Instances of properly extensive quantities are not necessarily infinitely divisible (as Lemma 1, in section 4.1, below, shows). Russell (1903) uses the term in a completely different way. He treats divisibility as a quantity *itself*, where short lines are less divisible than longer ones, which are less divisible still than two-dimensional regions, and so on. His (1903) is also the first place I have found advocating that the term 'extensive' not be taken to entail divisibility.

[12]The 'properly' modifier is meant to suggest, as I think is true, that this feature better characterizes the intuitive notion of *extension*, *being extended*, or *measure of extent* than the currently accepted sense of 'extensive' in terms of additivity. I won't offer a defense of this claim here.

[13]The conditions I give for properly extensive quantities in Perry (2015) make use of primitive ordering and summation relations between Q-properties. The M-R account has no such relations at the fundamental level. As such, if we want to take volume's proper extensiveness as fundamental, we need to express the constraints it puts on the parts of voluminous objects a different way. (Note that (5) and (6) will be of no help here. Once we plug in the M-R definitions for 'LESS-Q' and 'Q-SUM', they become instances of the trivial '$\Box(P \leftrightarrow P)$'). In section 3.6, I show

else, we can show that the volume ordering and summation relations, as defined by the M-R account, and the volume ratio relations, whose M-R definitions are generated by the "ratio procedure" (formally defined in section 4.2), are faithfully represented by the arithmetical ordering, summation, and ratio relations on the real numbers. Section 5 concludes and clarifies some issues set aside in the previous sections. Before presenting the formal M-R account of volume, it will be useful to understand how and why its competitors end up committed to quantitative structure, in particular metric structure, being radically extrinsic.

2 The Extrinsicality Worry

The M-R account defines volume metric relations in an intrinsic way. I have claimed that this is the result we should want. That is, insofar as we take quantitative structure to explain (or be part of the explanation for) physical phenomena, we should, thereby, want our account of what that structure *is* to render it (or the relevant sub-structure) *intrinsic* to the systems it's called upon to help explain. Consider, for instance, a cruel twist on the iced coffee scenario from the introduction: On this variant, the straight path from my body to the desk *is* shorter than my arm, but the ratio of the path's length to my arm's is 0.96-to-1. Even though I am close enough to reach the iced coffee, my fingertips can only just brush the sides of the cup, and, so, it remains frustratingly out of my grasp. The M-R account locates the source of the explanatory power of the numerical ratio 0.96-to-1 in the intrinsic properties and mereological structures of the physical entities involved (or the regions they occupy). Other accounts of quantity fail to do justice to the intrinsicality intuition. Let me explain why:

I have mentioned before that most other accounts take ordering and/or summation relations (or an analogue) as primitive, posit some axioms that these relations obey, and use them to ground metric structure.[14] These accounts ground metric structure *holistically*, by appealing to representation and uniqueness theorems. These are theorems that say a given domain (like the set of all that quantity's instances), over which some relations (the primitive ordering and summation relations) are defined that satisfy certain axioms, can be well represented by some mathematical structure or structures.

These theorems naturally suggest a certain way of defining the length ratio between x and y. Specifically, x and y's lengths stand in a ratio of n-to-1 just in case they imply that any function from (equivalence classes of same-lengthed) lengthy objects to real numbers, which preserves

that we can articulate these constraints in terms of the fundamental posits of the M-R account—viz. mereological relations and the sharing of intrinsic determinate length/volume properties. In the formal appendix section B.4, I prove that this characterization is equivalent the one in Perry (2015).

[14]This is a sensible approach, since there's little prospect in taking *metric* structure as primitive in any economical way. "n-times the volume of" and "m-times the volume of", if construed as two-place relations between voluminous entities (i.e. *not* as relations between a pair of voluminous entities and a number), are substantively different relations, and we would have to posit distinct axioms for each such relation if we were to take them as primitive.

the ordering and summation of the domain, maps x and y to numbers that, respectively, stand in the *mathematical* ratio n-to-1. That is,

(7) x is n-times the length of $y \leftrightarrow$ For any function, f, from the set of lengthy objects to $\mathbb{R}$, if f is such that (for any lengthy a, b, and c) b is at least as long as $a \leftrightarrow f(a) \leq f(b)$, and c is as long as a and b put together $\leftrightarrow f(a) + f(b) = f(c)$, then $f(x) = n * f(y)$. And there exists at least one such function.

A definition based on this biconditional would, clearly, be radically extrinsic. It would make "n-times the length of" dependent on the properties of certain functions from the total domain of lengthy objects to the real numbers. We might hope that the physical facts appealed to in order to *prove* the representation and uniqueness theorems may be used to give us an intrinsic definition. However, problems arise because representation and uniqueness theorems prove that a domain is homeomorphic to a given mathematical structure by appealing to *global* properties of a domain. Specifically, in addition to assumptions about and the distribution of ordering and summation relations (or some other sub-metrical analogue) over those domains, they make "structural" assumptions about the domain *itself*—i.e. that it is well populated, or that it has sufficient variegation in which of that quantity's magnitudes are instantiated.

Let me give a concrete example. What sorts of definitions would be available to an account based on one of the measurement theoretic systems of Krantz et al. (1971)? Consider the definition we get from Krantz, et. al.'s definition of a function from objects to numbers used to prove a representation theorem about "Archimedean Ordered Local Semigroups", which means a domain of entities with an *ordering relation* (Ordered) and *summation operation* on them such that no lengthy object is infinitely longer than any other (Archimedean), and the "sum" operation '$\circ$' needn't be defined for *every* pair of objects of the domain (Local Semigroup). It's plausible that length is such a quantity:[15]

(8) $$x \text{ is } n\text{-times longer than } y \quad =_{df} \quad \lim_{m \to \infty} \frac{N(x_m, y)}{N(x_m, x)} = n$$

Where $n \in \mathbb{R}$, and the term '$N(x, y)$' denotes the maximum number of objects with the same length as x such that y is longer than the sum of their lengths, and $x_1, x_2, x_3, \ldots$ are an infinite sequence of lengthy objects whose lengths converge on 0m in the limit.

This definition makes appeal to entities whose existence is only ensured by certain "existence and richness" axioms on the domain. Specifically, Krantz, et al. use what they call a "solvability" axiom, which assumes that the domain contains an object that "solves" any "inequalities"

[15]Krantz et al. say that length is an "extensive structure", which are a specific kind of Archimedean ordered local semigroup. An extensive structure is such that the ordering relation is transitive and total, and the summation operation is associative, commutative, and (regarding the length ordering) monotonic. We will not need to consider these axioms in detail, since they are not the source of the account's troublesome extrinsicality.

between a given pair of its elements. So, if a is shorter than b, there's some c such that a is as long or longer than b and c put together. This axiom will be required to ensure that each of the smaller and smaller x_i's exist, and that they converge on 0m. Another existence axiom is required in order to ensure that enough "copies" (i.e. other objects with the same length) of each of the x_i's exist. Without them, the term $N(x,y)$ isn't guaranteed to denote the right number. Moreover, while the total domain of lengthy objects is expected to satisfy these axioms, Krantz et al. warn against thinking that this means they'd be satisfied by a subset. They write that "..an axiom such as solvability may be false if attention is restricted just to that subset of objects tested: the solution to some inequality or equation may lie outside that subset. In fact, we may have accepted solvability to begin with because of the fine grainedness of the *entire* object set." (Krantz et al., 1971, p. 28, my emphasis).[16]

It's easy to see, then, how an account that employs "global" structural assumptions which apply over the domain taken as a whole, but not necessarily to a given subset, could fail to give an intrinsic account of metric structure. There is no guarantee that there is enough structure in the subsystem consisting of just x, y, and their parts to recapture metrical ratio relations.

Second-order Extrinsicality Objection

Some, like Mundy (1987), have thought that the problem with these existence and richness assumptions about the domain comes from their *contingency*. Representation and uniqueness theorems rely on global structural assumptions that are not guaranteed to be satisfied by a given subdomain. But, so the worry goes, its possible that a given subdomain have been *all that*

[16] I think that there's a kernel of a good definition here. That is, I expect Krantz et al.'s definition could be modified to give one equivalent to the version I defend in section 4. Imagine a variant of Krantz et al.'s account that adds the assumption that length is properly extensive, call it "$K+PE$" (for "Krantz plus proper extensiveness"). One (though certainly not the only) way to achieve this would be by adding the mereological axioms I present in section 3.6. The definition of length ratio relations on $K+PE$ is the same, except that the limit involved in (8) is replaced with:

$$\lim_{m\to\infty}\frac{N^*(a_m,y)}{N^*(b_m,x)}.$$

Where $a_1,a_2,a_3,\ldots$ are a sequence of *parts of* y whose lengths approach $0m$ in the limit, and likewise for $b_1,b_2,b_3,\ldots$ and x, *and* every a_n is as long as b_m if $n=m$. And the term '$N^*(x,y)$' is the equivalent of '$N(x,y)$' when you restrict your quantifiers to only y and y's parts.

The $K+PE$ does not need to rely on independent existence and richness assumptions about the domain. The existence of enough copies of each member of each sequence (the various a_i's and b_i's) are guaranteed by length's proper extensiveness. Since the constraints of proper extensiveness can be understood entirely in terms of x and y's parts, length ratio relations are intrinsic according to $K+PE$. However, while $K+PE$ improves on the original in some respects, it also comes with some added disadvantages. Specifically, many of $K+PE$'s axioms and primitive posits are *redundant*. That is, proper extensiveness plus the totality of the length ordering entails all the necessary axioms of the M-R account of length (just as it would for the M-R account of volume).

I argue that the M-R account for quantities like length can adequately define these metric relations (I *show it* for volume, below). If this is right, then we can give an adequate account of quantitative structure using only a few of $K+PE$'s axioms and only one of its two primitive relations (viz. length ordering). If we are interested in using proper extensiveness to give an intrinsic account in the spirit of Krantz et al.'s definition in the text, we would be much better off accepting the M-R account than we would $K+PE$. The M-R account is entailed by a proper subset of $K+PE$'s axioms, and employs no primitive quantitative relations.

there is. It seems to me, however, that this contingency is only a symptom of the broader problem of extrinsicality. We think the metric relations between elements of a given subdomain (of the lengthy objects, say) would have still obtained had that subdomain been all that there is *because* we think that length metric relations do not depend, for their instantiation, on anything outside their relata. This is an important result, if correct, since most accounts of quantitative structure which successfully avoid the contingency objection still render metric relations radically extrinsic.

For instance, Mundy (1987) posits primitive *second-order* relations of "ordering" and "summation", which relate mass[17] *properties*. He accepts a Platonism about properties according to which these universals, and the primitive second-order quantitative structural relations they stand in, are necessary existents. The first-order comparative mass relations between *objects* are all grounded in higher-order relations between their properties, which allows Mundy to avoid the contingency objection. Consider, for instance, his definition of "less massive than":

(9) x is less massive than $y =_{df}$ there exist mass universals U_1 and U_2 such that $U_1(x)$ and $U_2(y)$ and $U_1[<]U_2$ (where $[<]$ is the primitive second-order ordering relation).

No problem there. An instance of the primitive $[<]$ relation doesn't depend on anything, so its obtaining doesn't depend on things extrinsic to U_1 and U_2 – or x and y for that matter. If x being less massive than y depends on their intrinsic properties standing in a primitive two-place relations, then "less massive than" is not an extrinsic relation. However, when we move from the ordering relations to *metrical* relations, things don't look so good.

(10) x is n-times as massive as $y =_{df} \exists U_1, U_2($ U_1 and U_2 are mass universals, and... $U_1(x)$ and $U_2(y)$ and: (1) U_1 and U_2 are part of a domain of mass universals $\mathbf{M}$ such that the distribution of the primitive second-order ordering and summation relations over this domain satisfies axioms $A_1, A_2, A_3, \ldots$; (2) U_1 and U_2 are such that there's a function φ, from $\mathbf{M}$ to $\mathbb{R}$ – where for any universals $a, b, c \in \mathbf{M}$, $a[<]b$ iff $\varphi(a) < \varphi(b)$, and $ab[*]c$ iff $\varphi(a) + \varphi(b) = \varphi(c)$, and $\frac{\varphi(U_1)}{\varphi(U_2)} = n$).

Where '$[*]$' is the three-place second-order summation relation over the mass universals. Here this definition, again, just depends on universals and the fundamental ordering and summation relations they stand in. If universals are necessary existents, and if the axioms governing the primitive second-order relations over them are necessary, then this definition avoids any contingency worry we might have. However, this doesn't help at all with the problem of extrinsicality. The obtaining of a given metric relation between a and b will (in part) depend on

[17] A key feature of Mundy's second-order account is its generality, the account applies in the exact same way to any quantities which share the same structure (the so-called "unsigned scalar quantities", like length, volume, temperature (in Kelvin), etc.).

universals neither a, b, nor any of their parts instantiate, and on the primitive relations those universals stand in.

Field and Extrinsicality

The only account that comes close to avoiding extrinsicality is Field's. The part of his account which fares best is the theory of spatial (or spatiotemporal) distance. Intrinsicality is a bit different for a relational quantity like distance. We shouldn't think of facts about the distance from a to b as needing to be intrinsic to a and b. Rather, we should think of them as a matter of being intrinsic to (the shortest) straight path from a to b. If this is right, then Field's definition of "the distance from x to y is twice that from z to w" (which we'll express as 'xyR_2zw') in terms of betweenness and congruence *does* satisfy the intrinsicality condition:

(11) $xyR_2zw \leftrightarrow \exists u(u \text{ is a point} \wedge u \text{ is between } x \text{ and } y \wedge xu\textsc{Cong}uy \wedge uy\textsc{Cong}zw)$

This relation between x, y, z and w is intrinsic to the straight lines xy and zw. It holds in virtue of the existence of a part, u, of xy, and the fundamental congruence relation, 'Cong', between zw and some parts of xy. This definition of "R_n" is only available for rational n; irrational metric relations (like "the distance from x to y is π-times the distance from z to w") are more difficult to define on this account. However, there's at least some reason to believe that these will be either intrinsic or, at least, not *radically* extrinsic.[18]

Unfortunately, Field's success does not extend to monadic quantities like length, mass, volume, or temporal duration.[19] Field (1980) describes how to extend his account to apply to scalar quantities: replace the spatiotemporal "betweenness" and "congruence" relations with "SC-betweenness" and "SC-congruence"[20] ('SC' for scalar). However, the analogue of Field's definition schema using these relations does not avoid the extrinsicality problem. That is, the scalar analogue of (11),

(12) $xyV_2zw \leftrightarrow \exists u(u \text{ is a voluminous body} \wedge u \text{ is SC-between } x \text{ and } y \wedge xuCuy \wedge uyCzw)$.

[18] There's a way to get closer to a general account of ratios, though it falls short of a definition. Field (1980) describes the comparison of products, $|x*y| < |z*w|$, which amounts to the comparison of ratios $\frac{x}{z} < \frac{w}{y}$ (where x, y, w, and z are either spatiotemporal distances or intervals of difference according to some scalar quantity).

[19] The proponent of Field's account would likely claim that the spatiotemporal scalars (length, volume, temporal duration) can be grounded in the right kind of distance facts. However, this does not yet guarantee that this grounding story will equip us with an intrinsic definition of these quantity's metric relations. Moreover, Maudlin (1993) has argued that, even in spaces where there are points, there's good reason to not take distance to be a fundamental quantity (though some doubt about these arguments have been expressed by Dees (2015)). Even putting those issues aside, if it turns out that space is gunky, and lacks points, it would be very implausible that distance is the fundamental spatiotemporal quantity.

[20] If congruence is analyzed as the sharing of intrinsic properties, then positing a distinct "SC-congruence" will be unnecessary. Cf. section 3.2

is not an intrinsic relation. This is because, while the spatiotemporal relation "between(yxz)" entails that x is a part of physical straight line yz, its scalar analogue "SC-between(yxz)" merely indicates that $y \leq x$ and $x \leq z$ where $\leq$ is that quantity's ordering relation. With no guarantee that x is part of either y or z, the relation $V_2(xy)$ according to the scalar version of definition 11 will *not* be intrinsic to x, y, or their fusion. The same will go for an application of this definition schema to other scalar quantities, like mass, temperature, length, or area.[21]

3 A Mereological-Reductive Theory of Volume

In this section and the next, I present a formal M-R account of the quantitative structure of *spatial volume*,[22] and show how this account generates definitions of the volume ratio relations. The M-R definitions avoid appeal to mathematical entities or to material entities outside of the relata and their parts. I will highlight the importance of volume's proper extensiveness in this theory, and make it clear how analogous M-R accounts for other properly extensive quantities can be constructed.

3.1 Mereology

This system assumes the axioms of classical extensional mereology (CEM).[23]

$$
\begin{aligned}
(P1)\quad & Pxx \\
(P2)\quad & (Pxy \wedge Pyz) \rightarrow Pxz \\
(P3)\quad & \neg Pyx \rightarrow \exists z(Pzy \wedge z \neq y \wedge \neg Ozx) \\
\text{(Sum)}\quad & \exists z(z \in S) \rightarrow \exists x(\forall w(w \in S \rightarrow Pwx) \wedge \forall w(Pwx \rightarrow \exists y(y \in S \wedge Owy))) \\
Oxy \quad & =_{df}\ \exists z(Pzx \wedge Pzy) \\
Cxyz \quad & =_{df}\ Pxz \wedge Pyz \wedge \forall w(Pwz \rightarrow (Owx \vee Owy))
\end{aligned}
$$

'Pxy' reads "x is a part of y". According to this system, parthood is reflexive (P1) and transitive (P2). I also assume the principle of strong supplementation (P3),[24] and unrestricted composi-

[21] However, see note 19

[22] I discuss some of the complications involved with *spatio-temporal* (as opposed to purely spatial or purely temporal) quantities in section 5.2.

[23] I'm reasonably confident that it's possible to get everything we want from my system using, instead of CEM, a weaker mereology that Varzi (2014) calls "minimal mereology" (MM). MM replaces (P3) with *weak* supplementation—'$\forall x \forall y(PPxy \rightarrow \exists z(PPzy \wedge \neg Ozx))$'. Where '$PPxy$' stands for "$x$ is a proper part of y", i.e. '$Pxy \wedge x \neq y$'. The reason for this is that the other axioms in this system will ensure that *voluminous* objects are guaranteed to satisfy something equivalent to a restriction of (P3) (even if other objects don't). It would also not be difficult to get most of the results we need, including all of the definitions of volume ratio relations, using a restricted *composition* rule (limiting fusions to, say, contiguous spatial regions). Thanks to Achille Varzi for extremely helpful discussion and advice regarding this issue.

[24] The antisymmetry of parthood—i.e. $\forall x \forall y((Pxy \wedge Pyx) \rightarrow x = y)$—follows from these axioms.

tion: (Sum) says that for any (non-empty) set of objects, there exists an object, x, which is their mereological sum. The predicates 'Oxy' and '$Cxyz$' are to be read, respectively, as "x overlaps y" and "x and y compose z".

3.2 Shared Properties

On the M-R account, volume is a determinable quantity associated with a class of fully determinate magnitudes, i.e. intrinsic volume properties. Since each such property is a *fully* determinate way of having volume, an object can instantiate *at most one* volume property. Let's introduce the two-place predicate '$\approx$'. $x \approx y$ just in case x instantiates the same determinate volume property as y. It can be pronounced more simply as: "x has the same volume as y", or "x is as voluminous as y". Those who are uncomfortable embracing a realist conception of properties, or who are sympathetic to comparativism[25] about quantities, may accept a variant of my account on which '$\approx$' is not a derived relation, but an unanalyzed primitive two-place predicate.[26] The rest of my presentation will be consistent with either approach (that is, I will not need to make any appeals to volume properties outside of my use of '$\approx$').

Since the determinate volume properties exclude one another, if x instantiates a different volume determinate from y, it follows that x doesn't instantiate the same volume determinate as y. From this, and the symmetry and transitivity of identity, we can derive:

($\approx$ Sym) $\quad \forall x \forall y\, (x \approx y \rightarrow y \approx x)$

($\approx$ Trans) $\quad \forall x \forall y \forall z\, ((x \approx y \wedge y \approx z) \rightarrow x \approx z)$

I will pronounce '$x \approx x$' as "x is voluminous" (since $x \approx x$ just in case x instantiates a determinate volume property). From ($\approx$ Sym) and ($\approx$ Trans) we can derive a limited form of reflexivity: if x bears $\approx$ to anything, then x is voluminous, i.e.

($\approx$ Ref) $\quad \forall x (\exists y (x \approx y) \rightarrow x \approx x)$

[25]Comparativism, in the case of volume, is the view that the determinate magnitudes associated with the quantity are comparative volume relations rather than monadic volume properties. Russell (1903) distinguishes between the "relative" view (comparativism) and the "absolute" view (the view that a quantity's determinate magnitudes are monadic properties). Comparativism about quantity (in particular, mass) has been recently defended by Dasgupta (2013).

[26]The comparativist variant retains many of the advantages of my preferred view, with two notable exceptions: (1) the variant theory cannot derive ($\approx$ Sym) and ($\approx$ Trans), so has to take them as additional brute axioms; (2) Volume's ordering, summation, and ratio relations—which are all defined, partially, in terms of '$\approx$'—will not be *internal* relations. Internal relations, recall, are those which depend solely on the intrinsic properties of their relata. The quantitative volume relations on this variant will be intrinsic to the system *composed by* their relata (since they depend on the distribution of the primitive two-place '$\approx$' relation over that system), but they will not be *internal*, since '$\approx$', on this variant, is no longer defined in terms of sharing intrinsic properties. However, given her aversion to intrinsic volume properties in general, the comparativist is unlikely to see (2) as a great loss.

3.3 Combination Principle

Let me introduce the three-place predicate '$xy \circ z$', which stands for "x and y are put together in the right way and compose z", or "x and y concatenate to make z".[27] The definition of '$\circ$' differs between different quantities. For *voluminous* objects, all that is required for a and b to be put together in the right way is for them not to overlap:

$$ab \circ c \ =_{df} \ \neg Oab \wedge Cabc$$

The defined-up '$\circ$' predicate can be used to formulate axioms that apply to quantities like length or temporal duration just as well as they apply to volume. That is, while different properly extensive quantities will disagree about what's required to be "put together in the right way", they will agree about the overall structure of the axioms. Hence, one could straightforwardly adapt this system to apply to a quantity like length, temporal duration, or the invariant relativistic interval, simply by introducing a different definition for '$\circ$'.[28]

In general, a combination principle encodes the role of '$\circ$' or 'put together in the right way', however it's defined, in a broader account of that quantity's structure. When it comes to the *Volume Combination Principle*, or (V-Comb), we encode the role of '$\circ$' as it's defined for volume (above) in our account of volume's quantitative structure. That is (filling in '$\circ$'s definition): if a and b are voluminous, don't overlap, and compose c, then c is voluminous.[29]

(V-Comb) $$a \approx a \wedge b \approx b \wedge \neg Oab \wedge Cabc \ \rightarrow \ c \approx c$$

3.4 Sub-Metrical Quantitative Structure

The M-R account defines "a is less voluminous than b", or '$a \prec b$', and "a is at least as voluminous as b", or '$a \preceq b$' as follows:

(13) $$a \preceq b \ =_{df} \ \exists x(Pxb \wedge x \approx a)$$

(14) $$a \prec b \ =_{df} \ a \preceq b \wedge a \not\approx b$$

That is, a is less voluminous than b just in case they differ in volume and a has the same volume as one of b's parts. The M-R account defines "c is as voluminous as a and b put together" (or

[27] I will sometimes write this as '$\circ abc$', '$\circ(a,b,c)$', or '$a \circ b = c$'. The '=' in the latter formulation should not be interpreted as the identity relation.

[28] As well as adopting the appropriately restricted variant of the totality assumption discussed in section 3.5, below.

[29] In the general case, the combination principle is this:

(Comb) $$a \approx a \wedge b \approx b \wedge (a,b) \circ (c) \ \rightarrow \ c \approx c$$

"c's volume is the sum of a and b's volumes") as "there exists some $x \approx a$ and $y \approx b$ such that $xy \circ c$".

3.5 Totality

One reason to choose spatial volume as our example is that the ordering relation on voluminous objects is, plausibly, a *total* order. That is, if a and b are voluminous but don't have the same volume, then either a's volume is greater than b's or vice versa.

(Totality*) $$a \approx a \wedge b \approx b \rightarrow (a \preceq b \vee b \preceq a)$$

Put another way: for any voluminous a and b, a is either less voluminous than, more voluminous than, or of the same volume as b. ('$a \preceq b \vee b \preceq a$' is equivalent to '$a \prec b \vee b \prec a \vee a \approx b$'). Not all properly extensive quantities satisfy unrestricted totality. While all of them satisfy *some* form of a totality axiom, for some quantities their ordering is only total within certain subdomains.[30] The M-R account should be, and, indeed, *is*, applicable to those properly extensive quantities as well.

Expressed in the fundamental, mereological terms of the M-R account, the totality axiom satisfied by volume says:

(Totality) $$a \approx a \wedge b \approx b \rightarrow \exists x\,((Pxb \wedge a \approx x) \vee (Pxa \wedge b \approx x))$$

In prose: If a and b are voluminous, then either a has the same volume as some part of b or vice versa.

3.6 Proper Extensiveness

The axioms (Additivity) and (Properly Extensive) jointly characterize volume's proper extensiveness. I'll discuss them in turn.

(Additivity) $$a \approx a \wedge b \approx b \wedge ab \circ c \rightarrow \forall x \forall y \forall z \big(x \approx c \wedge yz \circ x \rightarrow (y \approx a \rightarrow z \approx b)\big)$$

(Additivity), takes a bit of unpacking. All properly extensive quantities are additive: if a and b concatenate to make c, then c's volume is the "sum" of a's and b's volumes. Importantly, if a and b's volumes sum to c's, then c's volume cannot be the sum of a's volume and some volume *other* than b's (just as $6 + 9 = 15$ means that 15 cannot be the sum of 6 and some *other* number $\neq 9$). Here's how this feature is encoded in the axiom (Additivity): If z is a voluminous object

[30] One such case is the invariant relativistic interval. I discuss how the totality axiom would be restricted for quantities of this sort in section 5.2.

composed of voluminous x and y, put together in the right way, then either $x \approx a$ and $y \approx b$, or vice versa (since a and b's volumes sum to c's volume), or *neither* x nor y share their volumes with a or b.

(Properly Extensive) $\quad a \approx a \wedge Pab \wedge b \approx d \rightarrow (a \approx b \vee \exists x \exists y(x \approx a \wedge y \approx y \wedge xy \circ d))$

(Properly Extensive) has two jobs (indeed, earlier drafts broke it up into two distinct axioms). If we take the M-R definitions on board, then (Properly Extensive) is equivalent to saying that (1) the $\preceq$ ordering on voluminous objects is transitive, and (2) whenever a is less voluminous than b, b is as voluminous as a and something else put together. If a is one of b's voluminous parts, then anything with the same volume as b, call it d must have a part with the same volume as a. If $a \approx b$, then this part is d itself; otherwise d is composed of a pair of voluminous parts, one of which has the same volume as a.

3.7 Within-object Archimedean assumption

Finally, I'll introduce an assumption that, while not necessary to obtain the results we want from this system, greatly simplifies our presentations of the definitions in the next section, and the proofs in the appendix. It amounts to the stipulation that there can be no voluminous entity which is infinitely more voluminous than some other one. More technically, it says that, if b is some voluminous entity, then b cannot be composed of an infinite set of non-overlapping parts, all with the same volume.

(Within-Object Archimedean)

$$b \approx b \wedge Pab \rightarrow \forall S \left(S = \{x \middle| Pxb \wedge x \approx a \wedge \forall y \neq x (y \in S \rightarrow \neg Oxy)\} \rightarrow S \text{ is finite} \right)$$

One interesting consequence of this Archimedean assumption is that there cannot be a "zero magnitude" of volume. If by "b has zero volume" we mean that $\forall a \forall c(ab \circ c \rightarrow c \approx a)$, then the Within-Object Archimedean assumption entails that any such b must not be voluminous (i.e. $b \not\approx b$).[31] This assumption, therefore, implies that points of space, one-dimensional lines, or two-dimensional planes in space are quite literally volume-less—they do not instantiate a volume magnitude. This doesn't mean that we deny that these entities *exist*; it just means we

[31] Why? Make the weakest version of the claim that $b \approx b$ has zero volume, i.e. that, for *some* a and c, $ab \circ c$ and $a \approx c$. From this supposition, it's easy to construct an infinite set of non-overlapping parts of c all with the same volume as b:

By (Properly Extensive), since $b \prec c$, and $c \approx a$, it follows that there exist some $b' \approx b$ and $x \approx x$ such that $xb' \circ a$. By (Additivity), since $c \approx a$ and $xb' \circ a$, it follows that $x \approx a$. But this is the very position we started out in! That is, from the assumption that c can be divided into two, non-overlapping, voluminous parts, a and b, where $a \approx c$, it follows that a can be divided into two non-overlapping, voluminous parts, x and b', where $b \approx b'$ and $x \approx a$. This means we can repeat this process indefinitely, applying (Properly Extensive) and then (Additivity) in the same way to get infinitely many parts all as voluminous as b.

deny that such entities are voluminous (and so aren't picked out by phrases like "so-and-so's voluminous parts"). I could have accepted a weakened Archimedean assumption that allows for a zero magnitude of volume, but it would add unnecessary complexity while making no difference to what the system can prove.[32] I discuss a more substantive way we might weaken (Within-Object Archimedean) in section 5.1.

4 The M-R account of Volume's Metric Structure

This section I define a general procedure which, given a voluminous pair, a and b, determines the M-R account's definition of the volume ratio relation that they stand in—i.e. the relation we describe with "the volume ratio of a to b is 1-to-n" or "b is n-times the volume of a" (for some real number, n). Neither the procedure, nor the definitions it generates, will require quantification over anything other than a, b, and their parts, and they will need to appeal only to mereological relations and/or '$\approx$'.

4.1 Definition of the "taking out" procedure

The first step will be to define a different procedure, which I call "taking x out of y" for some voluminous x and y, which tells you how many "copies" of x can "fit inside" y, and whether there's some remainder. The ratio procedure, we shall see, is defined in terms of repeated applications of this procedure

To "take x out of y" is to determine the maximum number of non-overlapping proper parts y can be partitioned into such that all (except, perhaps, one) of those parts bear $\approx$ to x. That is, whenever we take x out of y, for $x \not\approx y$, the procedure outputs a pair of entities: A part, r, of y such that $r \preceq x$, which we'll call the "remainder". The second is an integer (we'll call it the "count"), which is the cardinality of a particular set, S, such that (1) every member of S bears $\approx$ to x, (2) no member of S overlaps any other member, (3) y is the mereological sum of all the members of $S \cup \{r\}$.

We **take a out of b**, where $a \approx a$ and $b \approx b$, as follows: If $b \prec a$, then there are no parts of b which bear $\approx$ to a.[33] The output of this procedure is the integer 0, and the remainder is b. If $a \approx b$, then the output of this procedure is the integer 1 and there is no remainder. $b \approx a$ so b is the fusion of 1 copy of a without remainder. The third case, $a \prec b$, is the more interesting one:

[32] This would amount to adding an exception just for the zero magnitude: e.g. we replace (Within-Object Archimedean) with a disjunction stating (roughly) that, for every voluminous a and b such that Pab, either the set S (as defined in the original axiom) is finite *or* a is such that, for all voluminous x, if $xa \circ y$ then $x \approx y$. If we were to add this exception, no change would be required to the expression of any of the axioms. However, the definitions of the two procedures in the next section, as well as many of the derivations in the formal appendix, will require the inclusion of a "& so-and-so is not a zero-volume" qualifier at certain key steps. Cf. (Balashov, 1999) for some considerations for and against positing a zero magnitude.

[33] This is shown by Lemma 14, in the Appendix section B.3.

$a \prec b$. So there exists a part, a'_0, of b, such that $a \approx a'_0$. By (Properly Extensive), since $a \not\approx b$, there exists some x such that $a'_0 \circ x = b$ – Call it "d_1". By (Totality), either $d_1 \preceq a$ or $a \preceq d_1$. If $d_1 \preceq a$, stop. d_1 is the "remainder" of this procedure, and the "count" is 1.

If it's not the case that $d_1 \preceq a$, then $a \prec d_1$. So there exists a part, a'_1, of d_1 such that $a \approx a'_1$. By (Properly Extensive), since $a \not\approx d_1$, there exists some x such that $a'_1 \circ x = d_1$ – Call it "d_2". By (Totality), either $d_2 \preceq a$ or $a \preceq d_2$. If $d_2 \preceq a$, stop. d_2 is the "remainder" of this procedure, and the "count" is 2.

Continue this procedure for every d_n arrived at in this way. I.e.:

If $d_n \preceq a$, then stop. d_n is the "remainder" of this procedure, and the "count" is n.

If it's not the case that $d_n \preceq a$, then $a \prec d_n$. So there exists a part, a'_n, of d_n such that $a \approx a'_n$. By (Properly Extensive), since $a \not\approx d_n$, there exists some x such that $a'_n \circ x = d_n$ – Call it "d_{n+1}". By (Totality), either $d_{n+1} \preceq a$ or $a \preceq d_{n+1}$. If $d_{n+1} \preceq a$, stop. d_{n+1} is the "remainder" of this procedure, and the "count" is $n+1$.

From this definition, it's easy to see that taking a out of b has a defined output for any voluminous a and b.[34] This procedure is unique up to the volume of the remainder, and taking x out of y, where $x \approx a$ and $y \approx b$, has the same output (up to the volume of the remainder) as taking a out of b.[35]

When $d_n \approx a$, then a "goes evenly into" b—i.e. b can be partitioned into $n+1$-many non-overlapping parts, all $\approx a$. This brings us to our first Lemma, which says that whenever there is a minimal element,[36] u, every voluminous entity is the fusion of k non-overlapping parts all $\approx u$, where k is some integer.

Lemma 1. *If there exists a minimal element, call it u, – that is, if $\exists u \forall x(x \approx x \to u \preceq x)$ – then, for all voluminous b, the remainder d_n left after we take u out of b bears $\approx$ to u.*

Proof. Since d_n is part of the output of taking u out of b, it must be that this procedure terminated with d_n. Hence, by the definition of the procedure, $d_n \preceq u$. But, by the minimality of u, $u \preceq d_n$. Hence, by the definition of $\preceq$, $u \approx d_n$. □

[34] *Proof:* Suppose $a \approx a$ and $b \approx b$. By (Totality), either $a \prec b$, $a \approx b$, or $b \prec a$. In the latter two cases, the result is trivial. In the case where $a \prec b$, if the procedure terminates at the n'th step, then (by the definition of the procedure) b can be partitioned into $n+1$ many non-overlapping parts, n of which bear $\approx$ to a, and one we'll call "d_n". In that case n is the count and d_n the remainder output by this procedure. So, for $a \prec b$, taking a out of b can fail to have an output only if there's no step at which the procedure terminates. However, if the procedure never terminates, then there exists a set, S, of non-overlapping parts of b such that $\forall x(x \in S \to x \approx a)$ which is infinite. However, this is ruled out by the Within-object Archimedean axiom. So the procedure will eventually terminate.

[35] Lemmas 9 and 10, respectively (Appendix section B.2).

[36] Fun fact: We don't need to make a global claim to establish that there exists a minimal element. Because volume is properly extensive, it will suffice to show that there exists *some* voluminous entity which lacks any parts with different volume. I.e.: $\exists u(u \approx u \wedge \neg\exists x(P(x,u) \wedge x \approx x \wedge x \not\approx u))$ alternatively $\exists u(u \approx u \wedge \forall x(P(x,u) \to (x \approx x \to x \approx u)))$.

4.2 Volume Ratio

We use the procedure for "taking x out of y" to define the volume ratio relations—i.e. those designated by statements like "b is n-times the volume of a" for some $n \in \mathbb{R}$ and voluminous pair a, b. We will define a "ratio procedure" which, as I mentioned before, will consist of repeated application of the taking out procedure: first taking a out of b and then, if there's a remainder, taking that remainder out of a, and so on. Each application of the taking out procedure gets us a better and better approximation of the ratio of a to b.

After defining this procedure I show how it allows us to generate the M-R account's definitions of volume ratio relations, and I'll argue that the relations picked out by this procedure are ratio relations properly-so-called. The relations themselves will not require appeal to, or quantification over, numbers or other mathematical objects. The M-R definition *will* make use of nonnegative integers, but only in the case where they serve to count the members of some well specified, finite class of voluminous entities.

4.2.1 The Ratio Procedure

This procedure consists of repeated applications of the "taking out" procedure. We construct a list of integers $K_{(a,b)} = \langle k_0, k_1, k_2, \ldots, k_i, \ldots \rangle$, which need not be a finite list. Each successive entry, k_i, in the list $K(x, y)$ is determined by the "count" output by each application of this procedure, as defined above. The "remainder" output by the i-th "taking out" procedure is used to indicate whether the list should continue after its i-th member, and, if it should, then that remainder also serves as one of the inputs for the next application of that procedure.

We want to find the volume ratio between $a \approx a$ and $b \approx b$. To do this, we perform the **ratio procedure** on the ordered pair $\langle a, b \rangle$, which generates an ordered list, $K(a, b) = \langle k_0, k_1, k_2, \ldots, k_i, \ldots \rangle$ (where $k_1, k_2, \ldots, k_i, \ldots \in \mathbb{Z}^+$ and $k_0 \in \mathbb{Z}^+ \cup \{0\}$), as follows:

0. If $a \approx b$, then taking a out of b yields a count of 1 and no remainder. In that case set $k_0 = 1$ and stop. k_0 is $K(a, b)$'s first and final entry. If $a \not\approx b$, proceed to step 1.

1. Take a out of b. This procedure will output a count, $f \in \mathbb{Z}$, and a remainder, call it 'r_1'. By the definition of this procedure, $r_1 \preceq a$, if it exists (since, if not, the procedure would not terminate at r_1).

1-(i): If $r_1 \approx a$, then set $k_0 = f + 1$ and stop. k_0 is $K(a, b)$'s first and final entry.

1-(ii): If $r_1 \not\approx a$ then $r_1 \prec a$. In that case, set $k_0 = f$ and proceed to step 2.

2. Take r_1 out of a. This procedure will output a count, $g \in \mathbb{Z}$, and a remainder, call it 'r_2'. By the definition of this procedure, $r_2 \preceq r_1$.

2-(i): If $r_2 \approx r_1$, then set $k_1 = g + 1$ and stop. k_1 is $K(a, b)$'s second and final entry.

2-(ii): If $r_2 \not\approx r_1$ then $r_2 \prec r_1$. In that case, set $k_1 = g$, and proceed to step 3.

In the general case, the N-th step of the construction of $K(a,b)$ is:

N. Take r_{n-1} out of r_{n-2}. This procedure will output a count, $h \in \mathbb{Z}$, and a remainder, r_n. By the definition of this procedure, $r_n \preceq r_{n-1}$.

N-(i): If $r_n \approx r_{n-1}$, then set $k_{n-1} = h + 1$ and stop. k_{n-1} is $K(a,b)$'s n-th and final entry.

N-(ii): If $r_n \not\approx r_{n-1}$ then $r_n \prec r_{n-1}$. In that case, set $k_{n-1} = h$, and proceed to step $N+1$.

There is no guarantee that the ratio procedure will end for any given a and b. However, this procedure is explicitly defined and so can be used to generate a determinate ordered list, $K(a,b)$, of integers. The list $K(a,b)$ is, therefore, defined for any voluminous a and b.[37] In the cases where this procedure does terminate, it also outputs a "final remainder" r_{final}.

4.2.2 Significance

Observe that, in cases where this procedure terminates, we have a perfect approximation. That is, a and b are both fusions of some integer number of non-overlapping parts all $\approx r_{final}$. Let's call these integers p and q, respectively. This means that we can characterize how much more voluminous b is than a by comparing how many different "copies" of r_{final} can "fit" in each. That is, the ratio of b to a is represented by $\frac{q}{p}$.

Let 'VRAT:$n(x,y)$' be the two-place relation we attribute to x and y when we say "x is n-times as voluminous as y". We now have a way to determine the ratio between b and a when the ratio procedure for a and b terminates: where p and q are the integers arrived at according to the process described in the last paragraph, then b is $\frac{q}{p}$-times as voluminous as y, i.e. VRAT:$\frac{q}{p}(b,a)$.

There is another way to arrive at $\frac{q}{p}$ using the list, $K(a,b)$, which doesn't appeal to r_{final}. Recall that $K(a,b) = \langle k_0, k_1, k_2, \ldots, k_n \rangle$, where each k_i is a non-negative integer (and is non-zero for $i \geq 1$). We can take these integers and use them to construct what is called a "simple continued fraction" of the form:

$$k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \ddots + \cfrac{1}{k_n}}}}$$

[37] Lemmas 11 and 12 in appendix section B.2 proves that the ratio procedure outputs a unique list, $K(a,b)$, for a given a and b, and that, for any $c \approx a$ and $d \approx b$, $K(a,b) = K(c,d)$.

We can write this more compactly as

$$k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\cdots\frac{1}{k_n}$$

For $a \leq b$ where the ratio procedure for a and b terminates, and $K(a,b) = \langle k_0, k_1, k_2, ..., k_n \rangle$,

$$\frac{q}{p} = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\cdots\frac{1}{k_n}$$

So the list $K(a,b)$, in cases where the ratio procedure for a and b terminates, can be used to characterize the ratio between a and b just as well as the remainder r_{final}. That is, it can also allow us to determine that $\text{VRAT}{:}\frac{q}{p}(b,a)$. This is good, because in the cases where the ratio procedure for a and b *doesn't* terminate, we do not have a final remainder, but we do have a (non-terminating) list $K(a,b)$.

In the non-terminating case, we can still use $K(a,b)$ to determine the ratio between a and b, despite the fact that $K(a,b)$ is an infinite list. In the cases where $K(a,b)$ is non-terminating, i.e. $K(a,b) = \langle k_0, k_1, k_2, ... \rangle$, then we will be able to construct what is called an "*infinite* simple continued fraction".

$$k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\cdots$$

Infinite simple continued fractions, it turns out, always converge on particular real numbers. In fact, one very cool feature of continued fractions is that, defined as I have done so, every positive real number can be uniquely expressed as a simple continued fraction.[38]

Each step of the ratio procedure gives us closer and closer approximations to the ratio between a and b. Since, in this case, it does not terminate, the ratio arrived at is the limit of this procedure. The number $r \in \mathbb{R}$ on which each successive step of these fractions converge is the analogue of our $\frac{q}{p}$ in the terminating case. This means that, when $K(a,b) = \langle k_0, k_1, k_2, ... \rangle$:

$$r = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\cdots$$

4.3 Volume Ratio Relations

We can now determine the general definition schema for the volume ratio relations. Recall that, b "partitions into" some class of its parts iff they are all voluminous, none of the members of that class overlap, and b is their fusion. The schema is as follows:

> $\text{VRAT}{:}n(b,a) =_{df} \exists r_1, r_2, \ldots \Big((b$ partitions into: k_0 parts which bear $\approx$ to a, and another part, $r_1) \wedge (a$ partitions into: k_1 parts which bear $\approx$ to r_1, and another part, $r_2) \wedge (r_1$ partitions into: k_2 parts which bear $\approx$ to $r_2) \wedge \ldots \Big)$

[38] Appendix section B.5 runs through the proof that any real can be expressed as a continued fraction, and that simple continued fraction expressions of positive real numbers are unique.

The right side of this definition is precisely the sufficient condition for the ratio procedure on $\langle a,b\rangle$ to output the particular list of integers $K(a,b) = \langle k_0, k_1, k_2, \ldots\rangle$. This definition only involves appeal to a, b and their parts, and, beyond the mereological relations, only appeals to '$\approx$', i.e. "instantiates the same determinate volume property as". Therefore, the volume ratio relations are intrinsic. Since every positive real number can be uniquely picked out by the simple continued fractions generated from a list $K(a,b)$, this definition allows us to associate ordered pairs of voluminous objects with a unique real number which characterizes their volume ratio. This schema is a formalized and mereologized version of the "*anthyphairetic ratio*" between some pair of objects, and the ratio procedure is closely related to the process of *anthyphairesis*.[39]

4.4 Representation Theorem

I have argued that we can, simply by counting up the right sets of their parts, match each ordered pair of voluminous objects to a unique real number. I've also suggested that there's good reason to think these numbers correctly characterize the physical volume ratio between that pair. The usual punch-line to an account of metric structure involves proving representation and uniqueness theorems. The M-R account of volume's metric structure, however, does not need to appeal to result of such a theorem to establish that there are volume ratio relations. Representation and uniqueness theorems are not necessary to give an account of the quantitative relations we appealed to in the explanations at the beginning of this paper.

We will prove representation and uniqueness theorems about this system, but *not* as part of our account of volume's metrical structure. Rather, it will show that the features of the volume ratio relations I point out above imply that these relations can be faithfully represented by the right mathematical ratios. As such, we will not need to prove the usual sort of theorem, that starts with only the ordering and summation relations over the domain of voluminous objects, and gets to the ratio relations by showing that functions from the domain to the real numbers which preserve ordering and summation all agree about certain metric facts.

We, on the other hand, can appeal directly to the volume ratio relations, whose physical definitions are fixed by the ratio procedure, and consider whether mappings from objects to numbers preserve volume's *ratio* structure.[40] As such, the M-R account grounds metric struc-

[39] Also called "antenaresis" or the Euclidean algorithm. The term '*anthyphairesis*' as the name of the process of reciprocal subtraction is from the Greek '*anthuphairein*' Cf. (Fowler, 1987, chap. 2).

[40] There is also a practical reason to move away from a representation theorem couched in terms of ordering and summation, which stems from the way I define volume ratios (viz. via a procedure which links voluminous pairs up to real numbers via continued fractions). The problem is this: Continued fractions are not amenable to even very simple arithmetic operations. As such, if we wanted to use the account of metricality to define a function from objects to numbers and then show that this function preserved ordering and summation structure, the proof would require an inordinate amount of complexity. Specifically, the problem is with summation structure. Continued fractions *do not like* being added to one another. Seriously, they *hate* it. Nobody even knew whether you could do it directly until someone came up with an algorithm for doing it simple enough to be performed (by a computer

ture in a thoroughly unit-free way. That is, we do not need to specify a particular function, φ, and some arbitrary voluminous object, u, to serve as the "unit" such that the image of any *other* voluminous object is defined in terms of the *end result* of the ratio procedure u and that object. Rather, we can express a simple rule which *any* function φ will satisfy just in case it faithfully represents/preserves volume's metric structure:

(**RULE**): If taking a out of b yields the count $k \in \mathbb{Z}$ and the remainder c, then $\varphi(b) = k * \varphi(a) + \varphi(c)$

What (**RULE**) does is show a correspondence between certain basic numerical relations and certain mereological ones. This is important because the definition of the ratio procedure for a given a and b is defined entirely in terms of repeated applications of the "taking out" procedure for various pairs of a's and b's parts. What this means is that the very ratio procedure defined in the previous section, combined with (**RULE**), will be able to provide a *full specification* of the numerical ratio between the numbers that a function must assign to a given voluminous pair, which is provably identical to the number which characterizes the volume ratio between that pair.

That is, this rule, while simple in expression, turns out to allow us to prove what I call the Direct Ratio Theorem for volume:

Direct Ratio Theorem. *Every function $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**) if and only if:*

$$\text{For all } a, b \in V,\ \text{VRAT:}n(b,a) \text{ iff } \varphi(b) = n * \varphi(a).$$

*Moreover, for any pair of functions φ and φ' which both satisfy (**RULE**), there exists some $m \in \mathbb{R}^+$ such that, for all $x \in V$:*

$$\varphi(x) = m * \varphi'(x)$$

Where m is such that, if there exists some $u, v \in V$ where $\varphi(v) = \varphi'(u)$, then $\text{VRAT:}m(u,v)$.

Rather than bothering with summation structure, this theorem concerns ratio structure directly. The proof of this theorem requires no postulation of an arbitrary unit. It concerns the feature which all such functions must have if they are to preserve the ratio structure of the voluminous entities.

In Appendix section A.2, I prove the Direct Ratio Theorem.

5 Conclusion

We want to understand what it is about the physical world that our mathematical representations pick out, and what it is about the world in virtue of which these representations

anyway) in *1972*, and it is still extremely complicated. (That someone is R.W. Gosper in: Gosper (1972). "Continued fraction arithmetic." HAKMEM Item 101B, MIT Artificial Intelligence Memo.)

are reliable. This is useful not just to our understanding of scientific practice, but also to a deeper understanding of the physical "quantitative structure" that we often engage with only via a mathematical surrogate. I've argued that, for properly extensive quantities, we can give a Mereological-Reductive account of their quantitative structure. This account is necessary, and gives reductive definitions of the relations which constitute that structure according to which those relations are *intrinsic*.

Here I'll clarify some points set aside during presentation of the formal M-R account for volume. I conclude with a discussion of the quantities left out by the M-R account. I outline how the view established in this paper can help us make strides towards an account of their quantitative structure.

5.1 Archimedean Assumption

One might object that this account *does* rely on a contingent assumption about the structure of the domain after all, since I assume that the world is *Archimedean*. However, there are two reasons this assumption is acceptable. The first is that the "Within-object Archimedean property" is still an intrinsic assumption. It says, roughly, that for any two voluminous a and b, there is always a finite number of non-overlapping copies of a that "fit" in b, and vice-versa. This basically amounts to the assumption that there are no pairs of voluminous objects that stand in what we might describe as an "infinite ratio". Since the focus of this paper is on metricality, it makes sense to simplify things for ourselves and rule this out.

However, this Archimedean assumption is not one that this system genuinely needs, even though it's a reasonable assumption to make about the actual world. That is, if we were to drop this assumption, we could still recapture many of the results of the view. The sort of "ratio" relations we would be able to define in the non-Archimedean case would correspond to something over and above what we think of as ordinary metric structure. The right representational tool would likely be some sub-structure of the *surreal* numbers. I think the M-R account could be extended in this way, though I won't argue for this in detail.

The way to go about it, I think, would be to define an equivalence relation over the quantities, such that each equivalence class contains all and only quantities which bear finite ratios to one another. We could, for instance, do this by way of something like the taking out procedure: for every $x \approx x$ and $y \approx y$, x and y are "*finitely comparable*" iff there exists an $n \in \mathbb{Z}^+$ such that *either* $x \preceq y$ and y can be partitioned into, at most, n non-overlapping copies with the same volume as x, *or* $y \preceq x$ and x can be partitioned into, at most, n non-overlapping copies with the same volume as y.

The Within-object Archimedean assumption will hold *within* each equivalence class carved out by this relation, and so the M-R account, unmodified, will apply to them. Ratios within equivalence classes, then, will be finite and defined in the normal way. Ratios between voluminous entities which are *not* finitely comparable would be infinite. We could just define "infinite ratio" to be the failure of finite comparability. Via the ordering we could define two kinds of infinite ratios (intuitively "infinitely-many-times *more* voluminous than" and "infinitely-many-times *less* voluminous than"). It's not clear if much else would need to be done to accommodate the non-Archimedean case, but my guess is that the M-R account of volume's quantitative structure has the resources to capture it.

5.2 Totality

Volume is a properly extensive quantity whose ordering satisfies an unrestricted totality condition. I mentioned above that there are quantities which do not satisfy totality. Consider, for instance, the case of the invariant relativistic interval, "$\boldsymbol{I}$", in special relativity. If we understand the interval as measuring something like the spatiotemporal "length" of a path through Minkowski space time, then the quantitative ordering relation is not total over the domain of all spatiotemporal paths. No space-like path, i.e. a path composed of events which are each at space-like separation from all of the others, is either shorter or longer than any time-like trajectory connecting two time-like separated events.[41] On the various ways $\boldsymbol{I}$ is represented, numerically, space-like paths are assigned negative (or imaginary) numbers, while time-like ones are assigned positive (or just real) numbers.

The ordering relation does apply, however, *within* each sub-domain (i.e. of all the time-like trajectories, or of all the space-like paths) and, indeed, the relation is total. So, in these cases, while $\boldsymbol{I}$ is, plausibly, a properly extensive quantity that does not satisfy (Totality) in general, there are analogues of the axiom which *are* satisfied by certain sub-domains. Within those sub-domains, ratio relations will be definable and faithfully representable with the right mathematical structure. These ratio relations will remain silent on the relationship between a time-like trajectory and a space-like path (since the ratio procedure for such a pair will be unperformable), but, in such a case, that's exactly what we want.[42]

5.3 Beyond Properly Extensive Quantities

On the whole, the alternate theories of quantity in the literature are more general than the M-R account, in that they apply to more quantities. The M-R account has many advantages,

[41] Indeed, even a classical version of spatiotemporal length would obey similar restrictions. In the classical space-time, paths which cross simultaneity slices (without doubling back) would have a spatiotemporal length measurable in units like seconds or years, while paths wholly contained within a slice have spatiotemporal length measured in meters, or feet. Within each of these domains the ordering is total, but there are no ordering relations between members of either domain—a 5 meter path is neither longer nor shorter than a 12 second one.

[42] The set of light-like trajectories pose an independent difficulty, since, on most numerical representations of $\boldsymbol{I}$, every such path is assigned $\boldsymbol{I} = 0$, despite the fact that, in a very real sense, proper sub-intervals of these paths are genuinely "shorter" than the paths of which they are a part. I think there are things to be said here, and an account of quantitative structure in terms of mereology will contribute greatly to our understanding of these issues, but a discussion of that here would take us too far afield.

but these advantages only extend as far as the properly extensive quantities. Indeed, there's no prospect to tweak the M-R account to generalize it, since the view crucially depends on a mereological feature that quantities like mass or temperature *do not have*. One might read this as a (perhaps defeasible) disadvantage of my account. I think this would be a mistake. Generality is a good-making feature of a theory insofar as we want to avoid giving an overly disjunctive account. However, a unified account is valuable only to the extent that it does not paper over metaphysically important distinctions. For the M-R account, the restriction to properly extensive quantities is not a handicap of the view. It's an explanation of *what it is* about these quantities that grounds their physical quantitative structure, and of what about them makes it such that this structure is faithfully represented by a given bit of mathematics. Losing the restriction to properly extensive quantities means losing the explanatory force of the M-R account. Indeed, the third chapter of my dissertation, "Additivity and Dynamics" outlines some very strong considerations against trying to apply the M-R account to mass in particular.

Moreover, I think the M-R account can help us make strides in the direction of an account of the structure of *non*–properly extensive quantities (i.e. merely additive or intensive quantities) as well. In the fourth chapter of my dissertation, "Problems for a Dynamic Theory of Quantity", I propose a theory of the quantitative structure of things like mass and charge in terms of their *dynamical* connections to other quantities, specifically the properly extensive ones. This sort of hierarchical theory depends on something like the M-R account to get off the ground, since it takes the quantitative structure of properly extensive quantities as given. So, in the case of mass, to say that that "x is twice as massive as y" is to say something, ultimately, about how similarly x and y react when they are impressed by forces—where the degree of similarity is grounded in the metric structure of velocity or acceleration (which are grounded in the properly extensive quantities length and temporal duration). This allows us to give an account of mass's metric structure in terms of the metric structure of properly extensive quantities.

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Mereology and Metricality

Formal Appendix

Z. R. Perry

This is the formal appendix to the paper titled "Mereology and Metricality". It comprises two parts. The first part, below, deals with the proving a representation and uniqueness theorem about the Mereological-Reductive account of volume presented in the main text (and glossed below). The second part, starting on page 44 with section B, deals with other interesting results, including some lemmas important to the proofs in the first part.

The M-R Account of Volume

Definitions:

Concatenation/"Put together in the right way": $ab \circ c =_{df}$ a and b don't overlap, and compose c.

Definition of $\leq$: $a \leq b =_{df}$ a bears $\approx$ to some part of b.

Definition of $\prec$: $a \prec b =_{df}$ a bears $\approx$ to some part of b and $a \not\approx b$.

Definition of "Sum": c is as voluminous as a and b put together (alternatively, c's volume is the sum of a's and b's volumes) $=_{df}$ there exists an $x \approx a$ and $y \approx b$ such that $xy \circ c$

Axioms:

(**V-Comb**): If a and b are voluminous, don't overlap, and compose c, then c is voluminous. (i.e. if $ab \circ c$, then $c \approx c$)

(**Additivity**): If $ab \circ c$, then, for any $x \approx c$, if x has parts y and z which don't overlap and compose it, then $y \approx a$ just in case $z \approx b$.

(**Properly Extensive**): If a bears $\approx$ to a part of b and $b \approx c$, then either $a \approx b$ or there exists an object, $x \approx x$, such that $ax \circ c$.

(**Totality**): If a and b are voluminous, then either a bears $\approx$ to some part of b or b bears $\approx$ to some part of a.

Simplifying Assumption:

Within-Object Archimedean Property: If a bears $\approx$ to a part of b, then every set, S, of non-overlapping parts of b such that $\forall x(x \in S \rightarrow x \approx a)$, is finite.

This system also presupposes the axioms of classical extensional mereology (CEM).

(Note: In the remainder of this paper, voluminous entities denoted by primed terms always bear $\approx$ to the denotation of the corresponding unprimed term. So $x \approx x'$, for any voluminous x.)

A The Direct Ratio Theorem

First, I'll specify a condition which all and only the acceptable functions, φ, from voluminous objects to the positive reals must satisfy. We'll write "Taking a out of b outputs a count of k and a remainder of r" as "$T.O.(a,b,k,r)$". Here is the condition on φ:[43]

RULE: *If* ($T.O.(a,b,k,r)$ or $T.O.(a,b,k,\emptyset)$) *then* $\varphi(b) = k * \varphi(a) + n$ where $k \in \mathbb{Z}$ and $n = \varphi(r)$ if there exists a remainder and 0 otherwise.

From this rule alone, it's very straightforward to prove what I call the *Direct Ratio Theorem*. It is a variant of the more familiar style of representation and uniqueness theorems. I call it a "ratio theorem" because it explicitly asserts the correspondence between the Vrat relation and the mathematical ratio between the numbers any φ assigns to voluminous objects (I use '$\psi : A \mapsto B$' to denote a function, ψ, from set A to set B).

Direct Ratio Theorem. *Every function $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**) if and only if, for all $a, b \in V$.*

(i) $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

(ii) Vrat:$n\,(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

*Moreover, for any pair of function φ and φ' which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such that, for all $x \in V$:*

$$\varphi(x) = m * \varphi'(x)$$

Where m is such that, if there exists some $u, v \in V$ where $\varphi(v) = \varphi'(u)$, then Vrat:$m\,(u,v)$.

A.1 Preliminary Lemmas

A.1.1 An Important Result

Lemma 2. *If $a \prec b$, where $a, b \in V$, then, for any $\varphi : V \mapsto \mathbb{R}^+$ that satisfies (**RULE**), $\frac{\varphi(b)}{\varphi(a)} > 1$.*[44]

[43] I will sometimes, for simplicity, refer instead to the weaker:

RULE*: *If* $T.O.(a,b,k,r)$ *then* $\varphi(b) = k * \varphi(a) + \varphi(r)$ (where $k \in \mathbb{Z}^+ \cup 0$ and $r \preceq a$).

Suppressing the case where there is no remainder, (since that is just the special case where $a \approx b$) the simplified version (**RULE***) is equivalent to the more complicated (**RULE**) when it is combined with the stipulation that $a \approx b \Leftrightarrow \varphi(a) = \varphi(b)$.

[44] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, 59.

Proof. Pick some arbitrary $a, b \in \mathbf{V}$ and an arbitrary function, φ, that satisfies (**RULE**). Suppose $a \prec b$. By (**RULE**), then, $\varphi(b) = k * \varphi(a) + \varphi(r)$. So $\frac{\varphi(b)}{\varphi(a)} = k + \frac{\varphi(r)}{\varphi(a)}$. Since its numerator and denominator are positive reals, the fraction $\frac{\varphi(r)}{\varphi(a)} \in \mathbb{R}^+$, and, since $a \prec b$, $k \geq 1$. So their sum is > 1.

□

A.1.2 The Key Lemma

The next step is that we prove the following:[45]

Lemma 3. *For any voluminous a and b, and any $\varphi : \mathbf{V} \mapsto \mathbb{R}^+$ that satisfies (**RULE**), $K(a,b) = \langle k_0, k_1, k_2, \ldots \rangle$ just in case*

$$\frac{\varphi(b)}{\varphi(a)} = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \ddots}}}$$

That is,

$$K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle \text{ iff } \frac{\varphi(b)}{\varphi(a)} = k_0 + \frac{1}{k_1+} \frac{1}{k_2+} \frac{1}{k_3+} \cdots \tag{1}$$

Proof. Suppose that φ is a function from $\mathbf{V}$ to $\mathbb{R}^+$ that satisfies (**RULE**).

Consider some arbitrary voluminous pair $a, b \in \mathbf{V}$.

By (**RULE**), $x \approx y$ if and only if $\varphi(x) = \varphi(y)$. So, if $a \approx b$, then $K(a,b) = \langle 1 \rangle$ (by step 0 of the ratio procedure) and $\frac{\varphi(b)}{\varphi(a)} = 1$, since $\varphi(a) = \varphi(b)$.

Now assume $a \not\approx b$. Then there exists some

$$K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle$$

If there exists a $\frac{\varphi(b)}{\varphi(a)}$, then, by the arguments in Appendix section (B.5), $\frac{\varphi(b)}{\varphi(a)}$ is a real number with a unique simple continued fraction expansion (where the final denominator, if it exists, is ≥ 2). That is, there exists some non-negative integers $f_0, f_1, f_2, \cdots \in \mathbb{Z}$ such that

$$\frac{\varphi(b)}{\varphi(a)} = f_0 + \frac{1}{f_1+} \frac{1}{f_2+} \frac{1}{f_3+} \cdots \tag{2}$$

[45] The proof of this lemma directly depends on Lemma 2. For the full map of Lemma interdependence, see figure 4, p. 59.

It suffices to show that for every i such that k_i is defined, there exists an f_i such that $k_i = f_i$. We demonstrate this below:

1.

We perform the ratio procedure for a and b. The first step of this procedure is to take a out of b. This will output a count, $s \in \mathbb{Z}$, and a remainder which we'll call "r_0". By (**RULE**),

$$\varphi(b) = s * \varphi(a) + \varphi(r_0) \tag{3}$$

Either $r_0 \approx a$ or not. We reason by cases. First, suppose $r_0 \approx a$, then (by the definition of the ratio procedure) $k_0 = s + 1$. Also, by (**RULE**), $\varphi(r_0) = \varphi(a)$. Hence

$$\varphi(b) = (s+1) * \varphi(a) \tag{4}$$

Dividing both sides by $\varphi(a)$ yields

$$\frac{\varphi(b)}{\varphi(a)} = \frac{(s+1) * \varphi(a)}{\varphi(a)} \tag{5}$$

$$\frac{\varphi(b)}{\varphi(a)} = f_0 = s + 1$$

So, if $r_0 \approx a$, then $K(a,b)$ has one member and $f_0 = k_0 = s + 1$. Now suppose $r_0 \not\approx a$. Then $k_0 = s$. So

$$\varphi(b) = s * \varphi(a) + \varphi(r_0) \tag{6}$$

Dividing both sides by $\varphi(a)$ yields

$$\begin{aligned} \frac{\varphi(b)}{\varphi(a)} &= \frac{s * \varphi(a) + \varphi(r_0)}{\varphi(a)} \\ &= s + \frac{\varphi(r_0)}{\varphi(a)} \\ &= s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} \end{aligned} \tag{7}$$

$T.0.(a,b,s,r_0)$ just in case $r_0 \preceq a$. Since $r_0 \not\approx a$, then $r_0 \prec a$. By Lemma 2, $\frac{\varphi(a)}{\varphi(r_0)} > 1$. Hence, s is the greatest integer less than $\frac{\varphi(b)}{\varphi(a)}$. So $s = f_0$. So in this case, as well, there exists a k_0 and $k_0 = f_0$. Since $k_0 = f_0$ in both cases, $k_i = f_i$ for $i = 0$.

2.

We proceed to step 2 of the ratio procedure and take r_0 out of a. Hence, when we take r_0 out of a, the procedure will output a count, $t \in \mathbb{Z}^+$, and a remainder $r_1 \leq r_0$. By (**RULE**),

$$\varphi(a) = t * \varphi(r_0) + \varphi(r_1) \tag{8}$$

Either $r_1 \approx r_0$ or not. We reason by cases. First, suppose $r_1 \approx r_0$. So $k_1 = t + 1$, and $K(a,b) = \langle s, t+1 \rangle$. Also by (**RULE**), $\varphi(r_1) = \varphi(r_0)$. Hence

$$\begin{aligned} \varphi(a) &= t * \varphi(r_0) + \varphi(r_0) \\ \varphi(a) &= (t+1) * \varphi(r_0) \end{aligned} \tag{9}$$

Dividing both sides by $\varphi(r_0)$ yields..

$$\frac{\varphi(a)}{\varphi(r_0)} = \frac{(t+1) * \varphi(r_0)}{\varphi(r_0)} \tag{10}$$

$$\frac{\varphi(a)}{\varphi(r_0)} = t + 1 \tag{11}$$

Which means that

$$\begin{aligned} \frac{\varphi(b)}{\varphi(a)} = s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} = s + \frac{1}{t+1} \\ \frac{\varphi(b)}{\varphi(a)} = f_0 + \frac{1}{f_1} = k_0 + \frac{1}{k_1} = s + \frac{1}{t+1} \end{aligned} \tag{12}$$

(Since $t \neq 0$, $t+1 \geq 1$. Hence $k_0 + \frac{1}{k_1}$ is a simple continued fraction whose final denominator is ≥ 2.) So, if $r_1 \approx r_0$, then $k_i = f_i$ for $i \leq 1$. Now suppose $r_1 \not\approx r_0$. Then $k_1 = t$ and $K(a,b) = \langle s, t, \ldots \rangle$. Recall,

$$\varphi(a) = t * \varphi(r_0) + \varphi(r_1) \tag{13}$$

Dividing both sides by $\varphi(r_0)$ yields..

$$\frac{\varphi(a)}{\varphi(r_0)} = \frac{t * \varphi(r_0) + \varphi(r_1)}{\varphi(r_0)}$$

$$\frac{\varphi(a)}{\varphi(r_0)} = t + \frac{\varphi(r_1)}{\varphi(r_0)} = t + \cfrac{1}{\cfrac{\varphi(r_0)}{\varphi(r_1)}} \tag{14}$$

By Lemma 2, $\frac{\varphi(r_0)}{\varphi(r_1)} > 1$, so the inverse is < 1, meaning t is the greatest integer less than $\frac{\varphi(a)}{\varphi(r_0)}$. Hence,

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= s + \cfrac{1}{\cfrac{\varphi(a)}{\varphi(r_0)}} \\ &= s + \cfrac{1}{t + \cfrac{\varphi(r_1)}{\varphi(r_0)}} \\ \frac{\varphi(b)}{\varphi(a)} &= s + \cfrac{1}{t + \cfrac{1}{\cfrac{\varphi(r_0)}{\varphi(r_1)}}}\end{aligned} \tag{15}$$

So in this case, as well, there exists a k_0 and a k_1 $k_0 = f_0 = s$ and $k_1 = f_1 = t$. Since $k_1 = f_1$ in both cases, $k_i = f_i$ for all $i \le 1$.

Since $r_1 \not\approx r_0$, then $r_1 \prec r_0$. We proceed to step 3 and take r_1 out of r_0. This procedure will output a count $u \in \mathbb{Z}$, and a remainder $r_2 \preceq r_1$... (and so on).

In the general case:

Suppose that $k_i = f_i$ for any $i \le n-2$. Suppose that $K(a,b)$ has at least $n-1$ members (i.e. k_{n-2} is not $K(a,b)$'s final member). We show that $k_{n-1} = f_{n-1}$.

By the definition of the ratio procedure, there exits remainders r_j for all $j \le n-1$, and $r_{n-1} \not\approx r_{n-2}$. Since $r_{n-1} \not\approx r_{n-2}$, we proceed to step N and take r_{n-1} out of r_{n-2}. This procedure will output a count, $v \in \mathbb{Z}$, and a remainder $r_n \preceq r_{n-1}$. By (**RULE**),

$$\varphi(r_{n-2}) = v * \varphi(r_{n-1}) + \varphi(r_n) \tag{16}$$

Either $r_n \approx r_{n-1}$ or not. We reason by cases. First, suppose $r_n \approx r_{n-1}$, then $k_{n-1} = v + 1$. Also, this means $\varphi(r_n) = \varphi(r_{n-1})$. Hence

$$\begin{aligned}\varphi(r_{n-2}) &= v * \varphi(r_{n-1}) + \varphi(r_n) \\ &= v * \varphi(r_{n-1}) + \varphi(r_{n-1}) \\ &= (v+1) * \varphi(r_{n-1})\end{aligned} \tag{17}$$

Dividing both sides by $\varphi(r_{n-1})$ yields..

$$\begin{aligned}\frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= \frac{(v+1) * \varphi(r_{n-1})}{\varphi(r_{n-1})} \\ &= v + 1\end{aligned} \tag{18}$$

Which means that $K(a,b) = \langle s, t, u, \ldots, v+1 \rangle$ and

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= s + \frac{1}{t+}\,\frac{1}{u+} \cdots \frac{1}{\dfrac{\varphi(r_{n-2})}{\varphi(r_{n-1})}} \\ &= s + \frac{1}{t+}\,\frac{1}{u+} \cdots \frac{1}{v+1}\end{aligned} \tag{19}$$

So, if $r_n \approx r_{n-1}$, then $K(a,b)$ has $n{-}1$ members and $\frac{\varphi(b)}{\varphi(a)} = f_0 + \frac{1}{f_1+}\,\frac{1}{f_2+} \cdots \frac{1}{f_{n-1}} = k_0 + \frac{1}{k_1+}\,\frac{1}{k_2+} \cdots \frac{1}{k_{n-1}}$.
Now suppose $r_{n-1} \not\approx r_{n-2}$. Then $k_{n-1} = v$. So

$$\varphi(r_{n-2}) = v * \varphi(r_{n-1}) + \varphi(r_n) \tag{20}$$

Dividing both sides by $\varphi(r_{n-1})$ yields..

$$\begin{aligned}\frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= \frac{v * \varphi(r_{n-1}) + \varphi(r_n)}{\varphi(r_{n-1})} \\ &= v + \frac{\varphi(r_n)}{\varphi(r_{n-1})} \\ \frac{\varphi(r_{n-2})}{\varphi(r_{n-1})} &= v + \frac{1}{\dfrac{\varphi(r_{n-1})}{\varphi(r_n)}}\end{aligned} \tag{21}$$

By Lemma 2, $\frac{\varphi(r_{n-1})}{\varphi(r_n)} > 1$, so the inverse is < 1, meaning v is the greatest integer less than $\frac{\varphi(n-2)}{\varphi(r_{n-1})}$.

Hence,

$$
\begin{aligned}
\frac{\varphi(b)}{\varphi(a)} &= s + \frac{1}{t+}\,\frac{1}{u+}\cdots\,\frac{1}{\dfrac{\varphi(r_{n-2})}{\varphi(r_{n-1})}} \\
&= s + \frac{1}{t+}\,\frac{1}{u+}\cdots\,\frac{1}{v+}\,\frac{1}{\dfrac{\varphi(r_{n-1})}{\varphi(r_n)}}
\end{aligned}
\tag{22}
$$

So in this case, as well, there exists a k_{n-1} and $k_{n-1} = f_{n-1} = v$. Since $k_{n-1} = f_{n-1}$ in both cases, $k_i = f_i$ for all $i \leq n-1$.

By induction on these steps, we can conclude that for any i, if k_i exists then f_i exists and $k_i = f_i$.

Since (1) is satisfied for an arbitrary voluminous pair a, b and an arbitrary function, φ, from voluminous entities to positive real numbers that satisfies (**RULE**), it follows that (1) is satisfied for any choice of a and b and for any φ satisfying (**RULE**). □

We have shown that Lemma 3 holds. Hence, for any voluminous $a \preceq b$ and any function φ from voluminous objects to real numbers that satisfies (**RULE**),

$$
K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle \text{ iff } \frac{\varphi(b)}{\varphi(a)} = k_0 + \frac{1}{k_1+}\,\frac{1}{k_2+}\,\frac{1}{k_3+}\cdots
\tag{1}
$$

Here is how we can proceed from Lemma 3 to the Direct Ratio Theorem. Recall that the definition of the various relations $\text{VRAT}{:}n\,(b,a)$ for any $a \preceq b$ was simply that $n = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\frac{1}{k_3+}\ldots$, where $K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle$. Hence Lemma 3 implies that, for any voluminous a and b, $\frac{\varphi(b)}{\varphi(a)} = n$, where n is the real number corresponding to the volume ratio relation such that $\text{VRAT}{:}n\,b, a$.

A.2 Proof of the Direct Ratio Theorem

Recall that the Direct Ratio Theorem says:

Direct Ratio Theorem. *Every function $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**) if and only if, for all $a, b \in V$.*

(i) $a \prec b$ iff $\varphi(a) \leq \varphi(b)$

(ii) $\text{VRAT}{:}n\,(b,a)$ iff $\varphi(b) = n * \varphi(a)$

*Moreover, for any pair of function φ and φ' which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such* *that, for all $x \in V$:*

$$\varphi(x) = m * \varphi'(x)$$

Where m is such that, if there exists some $u, v \in V$ where $\varphi(v) = \varphi'(u)$, then VRAT:$m(u,v)$.

Proof. We'll divide this theorem up into four lemmas, and prove each of those, occasionally relying on the result obtained in the proof of the previous lemma.

Lemma 4. *If $\varphi : V \mapsto \mathbb{R}^+$ satisfies (**RULE**), then it satisfies*

(i) $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

(ii) VRAT:$n(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

for all $a, b \in V$.[46]

From Lemma 3, for any voluminous pair a and b, $\frac{\varphi(b)}{\varphi(a)} = n$, where n is the real number corresponding to the volume ratio relation such that VRAT:$n(b,a)$.

Given this result, the proof for (ii) is trivial:

Proof. By Lemma 3, for any voluminous pair a and b, $\frac{\varphi(b)}{\varphi(a)} = n$ if and only if n is the real number corresponding to the volume ratio relation such that VRAT:$n(b,a)$. Since $\frac{\varphi(b)}{\varphi(a)} = n$ just in case $\varphi(b) = n * \varphi(a)$ (multiplying both sides by $\varphi(a)$), we can conclude that VRAT:$n(b,a)$ iff $\varphi(b) = n * \varphi(a)$. □

The proof for (i) is also quite simple:

Proof. Taking each direction of the biconditional in turn:

Left to Right: By the definition of the ratio relation, if VRAT:$n(x,y)$, then $n > 1$ just in case $y \prec x$. Because, the definition of the ratio relation says that $n = k_0 + \frac{1}{k_1+}\frac{1}{k_2+}\dots$ where $k_0, k_1, \dots$ are the first, second, etc. elements of the list $K(y,x)$. And, by the definition of the ratio procedure, $k_0 \neq 0$ and k_1 is defined iff $y \prec x$ (if $y \approx x$ then k_1 is not defined, and if $x \prec y$ then $k_0 = 0$).

Hence, for an arbitrary $a, b \in \mathbf{V}$, if $a \prec b$ then the n such that VRAT:$n(b,a)$ is > 1. By Lemma 3, $\frac{\varphi(b)}{\varphi(a)} > 1$, since $\frac{\varphi(b)}{\varphi(a)} = n$. So $\varphi(b) > \varphi(a)$.

Right to Left: We prove the contrapositive. For some arbitrary $a, b \in \mathbf{V}$. If $\frac{\varphi(b)}{\varphi(a)} \leq 1$, then, by Lemma 3, it couldn't be that VRAT:$n(b,a)$ for $n > 1$. Hence, by the definition of the ratio relation, $\neg(a \prec b)$.

□

Lemma 5. *If $\varphi : V \mapsto \mathbb{R}^+$ satisfies*

[46] The proof of this lemma directly depends on Lemma 3. For the full map of Lemma interdependence, see figure 4, p. 59.

(i) $a \prec b$ *iff* $\varphi(a) \leq \varphi(b)$

(ii) $\text{VRAT}{:}n(b,a)$ *iff* $\varphi(b) = n * \varphi(a)$

for all $a, b \in \mathbf{V}$*, then* φ *satisfies (**RULE**).*[47]

Proof. Let φ be some function from voluminous entities to positive real numbers such that (i) and (ii). It suffices to show that φ satisfies:

RULE: *If* $T.O.(a,b) \Rightarrow \langle k, d_k \rangle$ *then* $\varphi(b) = k * \varphi(a) + \varphi(d_k)$ (where $k \in \mathbb{Z}$ and $d_k \preceq a$).

Suppose $T.O.(a,b) \Rightarrow \langle k, d_k \rangle$. It follows that $a \preceq b$, so (by the ratio procedure) $K(a,b) = \langle k_0, k_1, k_2, k_3, \ldots \rangle$ is defined. Hence (by the definition of the ratio relations) $\text{VRAT}{:}n(b,a)$, where $n = k_0 + \frac{1}{k_1+} \frac{1}{k_2+} \frac{1}{k_3+} \ldots$. By (ii),

$$\frac{\varphi(b)}{\varphi(a)} = n = k_0 + \frac{1}{k_1+} \frac{1}{k_2+} \frac{1}{k_3+} \ldots$$

By the definition of the ratio procedure, k_0 is either the count, $c \in \mathbb{Z}$, output by taking a out of b, or $k_0 = c + 1$, in which case $a \approx r_0$. We reason by cases. Suppose $k_0 = c + 1$, hence $a \approx r_0$. By the definition of the ratio procedure, this means that k_0 is $K(a,b)$'s first and only member. By (ii), this means that

$$\frac{\varphi(b)}{\varphi(a)} = k_0 = c + 1$$

Hence

$$\varphi(b) = (c+1) * \varphi(a) = c * \varphi(a) + 1 * \varphi(a)$$

Since $a \approx r_0$ if and only if $\varphi(a) = \varphi(r_0)$, then $\varphi(b) = c * \varphi(a) + \varphi(r_0)$. Hence, if $k_0 = c + 1$, then φ satisfies (**RULE**).

Now suppose $k_0 = c$. Hence $r_0 \prec a$ or the remainder is null. If the remainder is null, then $c = 1$ and $a \approx b$. Hence $\varphi(a) = \varphi(b)$, so $\varphi(b) = 1 * \varphi(a)$. If $r_0 \prec a$, then $K(a,b)$ has at least two members.

Consider the list $K(a,b)$ and the list $K(r_0, a)$. It is clear from the definition of the ratio procedure that, the $(i+1)$'th element in $K(a,b)$ is identical to the i'th element of $K(r_0, a)$. That is, the members of $K(a,b)$, in order, starting with k_1 (i.e. excluding k_0), is identical to the ordered list of integers, $K(r_0, a)$.

[47] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

So, labeling the elements of $K(a,b)$ as 'k_i' and the elements of $K(r_0,a)$ as f_i, by (ii) it follows that:

$$\frac{\varphi(b)}{\varphi(a)} = n = k_0 + \frac{1}{k_1+}\dots$$

and

$$\frac{\varphi(a)}{\varphi(r_0)} = f_0 + \dots = k_1 + \dots$$

Hence

$$\begin{aligned}\frac{\varphi(b)}{\varphi(a)} &= k_0 + \frac{1}{\frac{\varphi(a)}{\varphi(r_0)}} \\ &= k_0 + \frac{\varphi(r_0)}{\varphi(a)}\end{aligned}$$

Multiplying both sides of the equation by $\varphi(a)$ yields:

$$\begin{aligned}\varphi(b) &= (k_0 + \frac{\varphi(r_0)}{\varphi(a)}) * \varphi(a) \\ &= k_0 * \varphi(a) + (\frac{\varphi(r_0)}{\varphi(a)}) * \varphi(a) \\ \varphi(b) &= k_0 * \varphi(a) + \varphi(r_0)\end{aligned}$$

Hence, if $r_0 \prec a$, then φ satisfies (**RULE**).

Since it can be shown for all cases, it follows that φ satisfies (**RULE**). Since we reached this conclusion from the assumption that φ satisfies (i) and (ii), it follows that any function φ which satisfies (i) and (ii) satisfies (**RULE**). □

Lemma 6. *If φ and ψ are both functions from* **V** *to the non-negative reals which satisfy (**RULE**), there exists some $m \in \mathbb{R}$ such that, for all $x \in V$:*[48]

$$\varphi(x) = m * \psi(x)$$

Proof. Suppose there exists some φ and ψ, both functions from the domain of voluminous objects to the reals. Suppose φ and ψ satisfy (**RULE**). Hence, by Lemma 4, φ and ψ satisfy (i) and (ii).

We want to show that, for all $x \in \mathbf{V}$, $\varphi(x) = m * \psi(x)$ for some m. Note that, since $\varphi(x) \in \mathbb{R}$

[48]The proof of this lemma directly depends on Lemmas 4 and 11. For the full map of Lemma interdependence, see figure 4, p. 59.

and $\psi(x) \in \mathbb{R}$, then clearly $\frac{\varphi(x)}{\psi(x)} \in \mathbb{R}$. As such, it will suffice to show that, for every $u, v \in \mathbf{V}$, $\frac{\varphi(u)}{\psi(u)} = \frac{\varphi(v)}{\psi(v)}$.

Pick an arbitrary voluminous pair, $a, b \in \mathbf{V}$. By Lemma 11, for every ordered pair there exists some ratio relation, call it 'VRAT:n', such that VRAT:$n(a,b)$. Since both φ and ψ satisfy (ii),

$$\varphi(a) = n * \varphi(b) \tag{23}$$

and

$$\psi(a) = n * \psi(b) \tag{24}$$

Hence,

$$\begin{aligned} \frac{\varphi(a)}{\psi(a)} &= \frac{n * \varphi(b)}{n * \psi(b)} \\ &= \frac{n}{n} * \frac{\varphi(b)}{\psi(b)} \end{aligned}$$

therefore

$$\frac{\varphi(a)}{\psi(a)} = \frac{\varphi(b)}{\psi(b)}$$

We have shown that $\frac{\varphi(a)}{\psi(a)} = \frac{\varphi(b)}{\psi(b)}$. Since it is true for an arbitrary $a, b \in \mathbf{V}$ it holds for any voluminous pair. Since $\frac{\varphi(a)}{\psi(a)} \in \mathbb{R}^+$, then there exists some $m \in \mathbb{R}^+$ such that, for any $x \in \mathbf{V}$, $\frac{\varphi(x)}{\psi(x)} = m$, i.e. $\varphi(x) = m * \psi(x)$. □

Finally, we show that,

Lemma 7. *Suppose that φ and ψ are functions from V to the non-negative reals such that*

$$\varphi(x) = m * \psi(x)$$

For all $x \in V$, then, if there exists some u, v such that $\varphi(v) = \psi(u)$, then m is the real number such that VRAT:$m(u,v)$.[49]

Proof. Given Lemma 6, It suffices to show that, for some $u, v, x \in \mathbf{V}$, if VRAT:$n(u,v)$ and $\varphi(x) = m * \psi(x)$, then $m = n$.

[49]The proof of this lemma directly depends on Lemmas 6. For the full map of Lemma interdependence, see figure 4, p. 59.

Suppose that $\textsc{Vrat}{:}n(u,v)$ and $\varphi(x) = m * \psi(x)$ for some $x \in \mathbf{V}$. By Lemma 6, $\varphi(x) = m * \psi(x)$ for *any* $x \in \mathbf{V}$. Since $u \in \mathbf{V}$, it follows that:

$$\frac{\varphi(u)}{\psi(u)} = m$$

Moreover, from (ii), we know that

$$\varphi(u) = n * \varphi(v)$$

Hence

$$n = \frac{\varphi(u)}{\varphi(v)}$$

However, since $\varphi(v) = \psi(u)$, it follows that

$$n = \frac{\varphi(u)}{\psi(u)}$$

Hence $n = \frac{\varphi(u)}{\psi(u)} = m$. So from the assumption that $\varphi(v) = \psi(u)$, it follows that $\textsc{Vrat}{:}m(u,v)$ (where $m = \frac{\varphi(x)}{\psi(x)}$ for any $x \in \mathbf{V}$). □

By Lemmas 4 and 5, every function φ from $\mathbf{V}$ to the non-negative real numbers satisfies (**RULE**) if and only if, for all $a, b \in \mathbf{V}$.

$$a \prec b \ \leftrightarrow\ \varphi(a) \leq \varphi(b)$$

and

$$\textsc{Vrat}{:}n(b,a) \ \leftrightarrow\ \varphi(b) = n * \varphi(a)$$

And, from Lemmas 6 and 7, if φ and φ' are functions from $\mathbf{V}$ to the non-negative reals which satisfy (**RULE**), then there exists some $m \in \mathbb{R}^+$ such that, for all $x \in \mathbf{V}$:

$$\varphi(x) \ = \ m * \varphi'(x)$$

Where m is such that, if there exists some $u, v \in \mathbf{V}$ where $\varphi(v) = \varphi'(u)$, then $\textsc{Vrat}{:}m(u,v)$.

Therefore, Lemmas 4, 5, 6, and 7 entail the Direct Ratio Theorem. □

B Additional Results and Useful Lemmas

This section contains proofs of various lemmas, some of which were explicitly appealed to in the main text or in the proof of the Direct Ratio Theorem, others of which are of independent interest. Sections B.1 and B.2 show that the two procedures presented in sections 4.1 and 4.2.1 of the main text are, indeed, functions from pairs of voluminous objects to their respective outputs. Section B.3 outlines a few lemmas appealed to in proofs elsewhere in the paper (in the main text as well as other parts of this appendix). Section B.4 connects the axioms I took to characterize proper extensiveness in section 3.6 in the main text to my presentation of them in Perry (2015). Section B.5 provides a discussion of continued fractions in case the reader is unfamiliar, and shows how each real numbers has can be uniquely expressed as a simple continued fraction.

B.1 The Taking-out Procedure is a Function

First we'll show, in the case of the taking-out procedure, that the output of that procedure is *defined* for any ordered pair of voluminous entities, $\langle a, b\rangle$, and is *unique* up to the volume of the remainder. Then we'll show that, given this result, the output of the ratio procedure is defined and unique for any ordered pair of voluminous entities.

Lemma 8. *For any voluminous pair a, b, taking a out of b always has an output, either $\{1\}$ or $\{k, r\}$ were $k \in \mathbb{Z}$ and $r \preceq a$ is a part of b.*[50]

The definition of the taking-out procedure, presented in Section 4.1, as well as the argument presented footnote 34 of the main text, amounts to a proof of this lemma.

Lemma 9. *The taking-out-of procedure for a voluminous pair is unique up to the volume of the remainder (that is, different remainders are possible for the same pair just in case they have the same volume).*[51]

Proof. Consider an arbitrary $a \approx a$ and $b \approx b$. We show that the procedure for taking a out of b is unique up to the volume of the remainder:

Uniqueness for the special cases—that is, where either $b \prec a$ or $b \approx a$—is built in: By the definition of the taking-out-of procedure, taking a out of b outputs a count of 1 and no remainder if *and only if* $b \approx a$. Similarly, by the definition of the procedure, taking a out of b outputs a count of 0 and a remainder of b if and only if $b \prec a$.

[50] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[51] The proof of this lemma directly depends on Lemmas 13, 14, and 15. For the full map of lemma interdependence, see figure 4, p. 59.

Since $b \prec a$, $a \approx b$, and $a \prec b$ are mutually exclusive (this follows from Lemma 14) and exhaustive possibilities for a voluminous pair, these biconditionals entail that the procedure for taking a out of b is unique for $a \approx b$ and $b \prec a$.

Now we show that the procedure is unique for $a \prec b$. Let $T.O.(a,b,n,r)$ be elliptical for "at least one performance of the procedure for taking a out of b results in a count of $n \in \mathbb{Z}$ and a remainder, r". Since $a \not\approx b$, there will always exist a remainder.

To prove uniqueness up to the volume of the remainder, it suffices to show that, for any voluminous pair $a \prec b$: If $T.O.(a,b,n,r)$, then, for any $m \in \mathbb{Z}$ and any voluminous part r^* of b, $T.O.(a,b,m,r^*)$ if and only if $n = m$ and $r \approx r^*$.

First, we'll show that, if $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $n = m$. Then we'll show that, if $n = m$, then $r \approx r^*$

We show that: $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*) \Rightarrow n = m$

Suppose, for reductio, that $n \neq m$, then either $n > m$ or $n < m$.

Suppose WLOG that $n < m$.

By the definition of the "taking out" procedure, n and m are the respective cardinalities of sets, S and S^*, of parts of b, where every member of the set bears $\approx$ to a, and no member of either set overlaps any other members of that set. $S^* = \{x_1, \ldots, x_m\}$. By Lemma 15, $fus(S) \approx fus(\{x_1, ..., x_n\})$. (I will use the term '$[.. \oplus ..]$' to represent the fusion of a pair of objects, and '$fus(X)$' to represent the fusion of all members of the set, X)

Since $fus(S^*)$ doesn't overlap r^*, $fus(\{x_1, ..., x_n\})$ and r^* don't overlap. No members of S^* overlap, so $fus(\{x_1, ..., x_n\})$ and $fus(\{x_{n+1}, ..., x_m\})$ don't overlap. Hence, $fus(\{x_1, ..., x_n\})$ and $[fus(\{x_{n+1}, ..., x_m\}) \oplus r^*]$ don't overlap and jointly compose b. By (Additivity), $fus(S) \circ r = b$ and $fus(S) \approx fus(\{x_1, ..., x_n\})$ imply that $r \approx [fus(\{x_{n+1}, ..., x_m\}) \oplus r^*]$.

Consider the fusion of $x_{n+1} \in S^*$ and r^*, call it "$[x_{n+1} \oplus r^*]$". $[x_{n+1} \oplus r^*]$ is a part of $[fus(\{x_{n+1}, ..., x_m\}) \oplus r^*]$. Since x_{n+1} and r^* don't overlap, $x_{n+1} \circ r^* = [x_{n+1} \oplus r^*]$ which, by (V-Comb), entails that $[x_{n+1} \oplus r^*]$ is voluminous.

Since $[x_{n+1} \oplus r^*]$ bears $\approx$ to a part of $[fus(\{x_{n+1}, ..., x_m\}) \oplus r^*]$ (viz. itself), and $r \approx [fus(\{x_{n+1}, ..., x_m\}) \oplus r^*]$, by (Properly Extensive), $[x_{n+1} \oplus r^*]$ bears $\approx$ to a part of r. Call this part "p".

By (Properly Extensive) again, x_{n+1} bears $\approx$ to a part, "x'_{n+1}", of p (since $p \approx [x_{n+1} \oplus r^*]$). $p \leq r$. If x'_{n+1} were $\approx r$, then, by (Properly Extensive), x'_{n+1} would have a part $\approx p$. But, by Lemma 14, if $x'_{n+1} \prec p$, then there are no parts of x'_{n+1} that bear $\approx$ to p. Hence $x'_{n+1} \not\approx r$.

But, since $x'_{n+1} \approx x_{n+1} \approx a$, a bears $\approx$ to a part of r. And, since $x'_{n+1} \not\approx r$, $a \not\approx r$. So $a \prec r$. But $T.O.(a,b,n,r)$, which means r is the remainder output of a performance of the "taking out of" procedure on a and b. But, by the definition of that procedure, a remainder is output only when the procedure terminates, and the procedure terminates (for $a \prec b$) just in case the remainder is $\preceq a$. Therefore because $T.O.(a,b,n,r)$, $r \preceq a$. But if $a \prec r$, then r cannot be $\preceq a$ (see Lemma 14). Contradiction!

Hence, if $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $n = m$.

Now we show that: $T.O.(a,b,n,r)$, $T.O.(a,b,m,r^*)$, and $n = m \Rightarrow r \approx r^*$

Suppose for reductio that $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ and $a \prec b$, but $r \not\approx r^*$.

By the previous argument, $n = m$. Let 'S' denote the set of n non-overlapping parts of b such that every member of S bears $\approx$ to a and $fus(S) \circ r = b$, and 'S^*' the set of m non-overlapping parts of b such that every member of S^* bears $\approx$ to a and $fus(S) \circ r^* = b$. By Lemma 15 and $n = m$, the fusion of all of S's members bears $\approx$ to the fusion of all of S^*'s members. That is $fus(S) \approx fus(S^*)$.

Since we've assumed that $r \not\approx r^*$, by (Totality), either $r \prec r^*$ or $r^* \prec r$. Suppose (WLOG) that $r \prec r^*$. Then there exists a part, x, of r^* such that $x \approx r$. By (Properly Extensive) and $x \not\approx r^*$, there must also exist a part, y, of r^* such that $x \circ y = r^*$.

By (Totality), either $r \prec r^*$ or $r^* \prec r$. Suppose (WLOG) that $r \prec r^*$. Then there exists a part, x, of r^* such that $x \approx r$. By (Properly Extensive) and $x \not\approx r^*$, there must also exist a part, y, of r^* such that $x \circ y = r^*$.

$fus(S^*)$ doesn't overlap r^* (or any of its parts). So $fus(S^*) \circ x = [fus(S^*) \oplus x]$ and, by (V-Comb), $[fus(S^*) \oplus x]$ is voluminous. By Lemma 13, from $fus(S) \approx fus(S^*)$, $r \approx x$, and $fus(S) \circ r = b$, it follows that $[fus(S^*) \oplus x] \approx b$.

However, since $x \circ y = r^*$, and $fus(S^*)$ and r^* compose b, it follows that $[fus(S^*) \oplus x] \circ y = b$. But, by (Within-Object Archimedean), $[fus(S^*) \oplus x] \not\approx b$. Contradiction!

So, $T.O.(a,b,n,r)$ and $T.O.(a,b,m,r^*)$ for some $a \prec b$, then $r \approx r^*$. □

Lemma 10. *If $T.O.(a,b,n,r)$ and $a \approx c$ and $b \approx d$, then $T.O.(c,d,n,r')$, where $r' \approx r$.*[52]

Proof. In the case where $a \approx b$ (and, therefore, $c \approx d$), this is trivial, since $n = 1$ and there is no remainder in both instances of the taking-out procedure.

Suppose $a \not\approx b$

[52]The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

Case 1: Taking a out of b has no remainder.

Suffices to show that: For all voluminous x, y, z, If x is composed of n-many non-overlapping parts, all bearing $\approx$ to y, then $z \approx x$ is composed of n-many non-overlapping parts, all bearing $\approx$ to y.

> Suppose x is composed of n-many non-overlapping parts all $\approx y$, $n \neq 1$. Let S be the set of $n-1$ of those parts, and consider the fusion $fus(S)$. Call the n'th part (not in S) y'. Since $fus(S)$ and y' don't overlap, $fus(S) \circ y' = x$.
>
> Since $z \approx x$, by (Properly Extensive), z has a part bearing $\approx$ to y'. Call it y''. By ($\approx$ Trans), $z \not\approx y'$ so, by (Properly Extensive), there exists some other part, w, of z such that $w \circ y'' = z$. By (Additivity), it follows that $w \approx fus(S)$.
>
> We have shown that, if x is composed of n-many non-overlapping parts all $\approx y$, then $z \approx x$ is composed of two non-overlapping parts: one $\approx y$, and one bearing $\approx$ to the fusion of $n-1$-many non-overlapping parts all $\approx y$. Induction on this result—plus (Within-Object Archimedean), which says that there are no voluminous fusions of infinitely many non-overlapping parts all $\approx y$—implies that z is the fusion of n-many non-overlapping parts all $\approx y$.

From this result and the transitivity of $\approx$, Lemma 10 follows for the case where taking a out of b has no remainder.

Case 2: Taking a out of b has a remainder, r.

> Suppose $T.O.(a, b, n, r)$. Let S be a particular set of n non-overlapping parts of b all bearing $\approx$ to a. It follows that $fus(S)$ is a part of b, and that $fus(S) \approx fus(S)$. By (Properly Extensive), since $b \approx d$, d has a proper part x such that $x \approx fus(S)$.
>
> By the definition of the procedure, $(fus(S), r) \circ (b)$. By (Within-Object Archimedean), $b \not\approx fus(S)$, hence $d \not\approx fus(S)$ so $d \not\approx x$. By (Properly Extensive), there exists a y such that $(x, y) \circ (d)$. By (Additivity), $x \approx fus(S)$ and $(fus(S), r) \circ (b)$ imply that $y \approx r$.
>
> By the definition of the procedure: $r \leq a$ and, by (Properly Extensive), $r \leq c$ since $a \approx c$. Since d can be partitioned into a pair of non-overlapping parts, one of which is the fusion of n-many non-overlapping parts all $\approx c$ and the other is a remainder $y \approx r$, then $T.O.(c, d, n, y)$.
>
> Therefore, $T.O.(c, d, n, y)$, where $y \approx r$, follows from the assumption that $T.O.(a, b, n, r)$, $a \approx c$, and $b \approx d$. So in the case where taking a out of b has a remainder, the conditional in 10.

Since it holds in the only two cases, it holds in general. □

What function can we associate with the taking-out procedure? The procedure takes two voluminous entities as inputs and outputs an integer and, usually, another voluminous entity. However, Lemmas 9 and 10 show that only these entities' volumes matter to the procedure.

So the function we associate with this procedure should map pairs of volume properties to pairs of integers and volume properties. That is, if $T.O.(a,b,n,r)$, and $V_x(a)$, $V_y(b)$, and $V_z(r)$, then the function should map $\langle V_x, V_y\rangle$ to $\{n, V_z\}$. We can define an analogous function in terms of '$\approx$' by replacing the volume properties with equivalence classes. In that case, the function maps pairs of equivalence classes, $\{x \mid x \approx a\}$ and $\{x \mid x \approx b\}$, to integer-equivalence class pairs, n and $\{x \mid x \approx r\}$.

B.2 The Ratio Procedure is a Function

Turning now to the ratio procedure. We show that the ratio procedure has a defined output for any pair (Lemma 11) and we show that that output is unique up to the volume of that pair (Lemma 12).

Lemma 11. *For any ordered pair of voluminous objects $\langle a,b\rangle$, there exists a list of non-negative integers $K(a,b) = \langle k_0, k_1, k_2, \ldots\rangle$ constructable via the ratio procedure on $\langle a,b\rangle$.*[53]

Given the existence of an output of the taking-out procedure for any ordered pair a and b Lemma 8, the definition of the ratio procedure in section 4.2.1 amounts to a proof of the existence of a $K(a,b)$ for any a and b.

Lemma 12. *If $K(a,b)$ is the list of integers output by the ratio procedure for a given $\langle a,b\rangle$, where a and b are voluminous, then, for any other ordered pair of voluminous objects $\langle c,d\rangle$ such that $a \approx c$ and $b \approx d$, there exists a list of non-negative integers, $K(c,d)$, constructable via the ratio procedure on $\langle c,d\rangle$ and $K(c,d) = K(a,b)$.*[54]

Proof. In addition to $K(a,b)$, the ratio procedure on $\langle a,b\rangle$ supplies us with a (possibly infinite) list of successive remainders, r_i, including an r_{final} just in case the procedure terminates. Each of these r_i's are parts of either a or b. Specifically, if $i = 2j$ for some $j \in \mathbb{Z}$, then r_i is part of a, and, if $i = 2j+1$, r_i is part of b.

Consider a token application of the ratio procedure on $\langle a,b\rangle$, where the list $R = \langle \ldots, r_i, r_{i+1}, \ldots\rangle$ of the successive remainders, denoted by 'r_i' for various $i \in \mathbb{Z}$. Recall that each remainder, r_i, is a particular voluminous object, that is a part of either a or b.

Also consider an application of the ratio procedure on $\langle c,d\rangle$, for which an entirely different list of voluminous objects, $R^* = \langle \ldots, r_i^*, r_{i+1}^*, \ldots\rangle$ are the successive remainders. Here the various r_i^*'s are either parts of c or parts of d.

[53] The proof of this lemma directly depends on Lemma 8. For the full map of Lemma interdependence, see figure 4, p. 59.

[54] The proof of this lemma directly depends on Lemmas 9 and 10. For the full map of Lemma interdependence, see figure 4, p. 59.

Step 1: Show that, for all i, $r_i \approx r_i^*$.

At the first step this is trivial: r_0 and r_{-1} are just a and b, respectively. Likewise $r_0^* = c$ and $r_{-1}^* = d$. By assumption $a \approx c$ and $b \approx d$, so $r_i \approx r_i^*$ for $i = -1$ and $i = 0$.

By Lemma 10, if $T.O.(r_n, r_{n-1}, x, r_{n+1})$ for some $x \in \mathbb{Z}$, then for any $r_n^* \approx r_n$ and $r_{n-1}^* \approx r_{n-1}$, it follows that $T.O.(r_n^*, r_{n-1}^*, x, r_{n+1}^*)$ and $r_{n+1}^* \approx r_{n+1}$. By Lemma 9, the $T.O.$ procedure is unique up to the volume of the remainder, so *any* part y of d such that $T.O.(r_n^*, r_{n-1}^*, x, y)$ must be $\approx r_{n+1}^*$ and, therefore, $\approx r_{n+1}$.

So, if $r_n \approx r_n^*$ and $r_{n-1} \approx r_{n-1}^*$, then $r_{n+1} \approx r_{n+1}^*$ so long as r_{n+1} and r_{n+1}^* are defined. Hence, since $r_{-1} \approx r_{-1}^*$ and $r_0 \approx r_0^*$, by induction it follows that $r_i \approx r_i^*$ for all $i \in \mathbb{Z}^+$ for which r_i exists.

Step 2: Show that the count output at each step is unique.

By the argument in Step 1, for any remainders r_i and r_i^* of two applications of the ratio procedure on $\langle a, b\rangle$ and $\langle c, d\rangle$, respectively, $r_i \approx r_i^*$. By Lemma 10, if taking r_i out of r_{i-1} outputs a count of $c \in \mathbb{Z}$, then taking r_i^* out of r_{i-1}^* will output a count of c if $r_i \approx r_i^*$ and $r_{i-1} \approx r_{i-1}^*$.

Hence, by this and Lemma 9 the count output at every step of the ratio procedure is unique, no matter which parts are involved in the procedure.

Consider the list $K(a, b)$ output by the ratio procedure on $\langle a, b\rangle$. Each entry, k_i, in this list is either the count output by taking some remainder r_i out of the previous remainder r_{i-1}, or (in the case of $K(a, b)$'s last entry, if it has one) it is that count +1. The value of k_i in $K(a, b)$, therefore, depends on the count output at step $i+1$, and on whether the procedure terminates at that step.

Step 3: Show that the ratio procedure on $\langle a, b\rangle$ terminate at the i'th step if and only if the ratio procedure on $\langle c, d\rangle$ does.

An application of the ratio procedure terminates at the i-th step *if and only if* either (1) there's no remainder (only possible in first step) or if (2) the i'th remainder bears $\approx$ to the previous remainder.

(1) is trivial. $c \approx a$ and $d \approx b$. If taking a out of b has not remainder at the first step, then $a \approx b$. By ($\approx$ Trans) and ($\approx$ Sym), it follows that $c \approx d$ and, hence, taking c out of d has no remainder.

Regarding (2), it suffices to show that if r_i, r_{i-1}, r_i^*, and r_{i-1}^* exist, then: $r_i \approx r_{i-1} \leftrightarrow r_i^* \approx r_{i-1}^*$.

Left to right: Suppose that these remainders exist and that $r_i \approx r_{i-1}$. By the argument in Step 1, for any remainder r_i from the ratio procedure on $\langle a, b\rangle$ and any remainder r_j^* from the ratio procedure on $\langle c,d\rangle$, if $i=j$ then $r_i \approx r_j^*$. Hence $r_i^* \approx r_i$ and $r_{i-1}^* \approx r_{i-1}$. By ($\approx$ Trans) and ($\approx$ Sym), it follows that r_i^*

By parallel reasoning, we can show the right to left direction of the biconditional.

Therefore, $k_i \in \mathbb{Z}$ is $K(a,b)$'s final entry if and only if k_i^* is $K(c,d)$'s final entry.

By the argument in Step 2, if neither k_i and k_i^* are their list's final entry, then $k_i = k_i^*$. Likewise, since the final entry is just the count +1, if both k_i and k_i^* *are* their list's final entry, then $k_i = k_i^*$.

Since all of their entries are the same, $K(a,b) = K(c,d)$. □

Hence, the ratio procedure associates an ordered pair of volume properties (or equivalence classes of voluminous entities) with a *unique* list of non-negative integers. For a given ordered pair of voluminous entities, it outputs the list of integers associated with the ordered pair of volume properties that those two, respectively, instantiate.

B.3 Other Useful Lemmas

This section contains three lemmas useful in simplifying proofs elsewhere in the paper. Lemma 13 is the "Unique Sum Lemma" in the following sense: if a given voluminous pair a,b concatenate to make some c, then any pair which instantiates the same volumes as a and b, respectively, must concatenate to make something with the same volume as c. That is,

Lemma 13. *If $(a,b)\circ c$, $(d,e)\circ f$, where $a\approx d$ and $b\approx e$, then $c\approx f$.*[55]

Proof. Suppose that $(a,b)\circ c$, $(d,e)\circ f$, $a\approx d$, and $b\approx e$. We'll show that $c\approx f$ by reductio:

That is, suppose $c \not\approx f$. By (Totality), either $c \prec f$ or $f \prec c$. Suppose WLOG that $c \prec f$. So f has a part, $c' \approx c$. By (Properly Extensive), there exists some part x of f such that $(c',x)\circ(f)$.

By (Properly Extensive) and $(a,b)\circ(c)$, c' has a part $a'\approx a$. By (Properly Extensive), there exists some part y of c' such that $(a',y)\circ(c')$. By (Additivity) and $a'\approx a$, $y\approx b$.

Since $(a',y)\circ c'$ and $(c',x)\circ f$, it follows that $(a',[x\oplus y])\circ f$, where $[x\oplus y]$ is the fusion of x and y. $a'\approx d$ (since $a\approx d$), so, by $(d,e)\circ f$ and (Additivity), $[x\oplus y]\approx e$.

However, $(y,x)\circ[x\oplus y]$ (since x and y are voluminous and don't overlap), which, by (Within-Object Archimedean), means $[x\oplus y]\not\approx y$. But $y\approx b$ (from above), and $b\approx e$! So, by ($\approx$ Trans), $[x\oplus y]\approx e$ and $[x\oplus y]\not\approx e$. Contradiction!

Therefore $c\approx f$. Hence, if $(a,b)\circ c$, $(d,e)\circ f$, $a\approx d$, and $b\approx e$, then $c\approx f$. □

The '$\preceq$' relation is, by definition, reflexive (since both the '$\approx$' relation and the parthood relation are reflexive). I said, on page 18, that (Properly Extensive) establishes the transitivity

[55]The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

of $\prec$. The transitivity of $\preceq$ follows from ($\approx$ Trans) plus the transitivity of $\prec$. Lemma 14, below, is the last step in establishing that the '$\preceq$' relation induces a total order on the set of equivalence classes determined by $\approx$. That is, it shows that the "X has an element which bears $\preceq$ to an element of Y" is an antisymmetric relation between sets of same-volumed objects. Put more simply, it says that:

Lemma 14. *If $b \preceq a$ and $a \preceq b$, then $a \approx b$.*[56]

Proof. If $x \prec y$ then $x \preceq y$. Hence, it suffices to show that '$\prec$' is *a*symmetric. That is, that $b \prec a$, then $\neg(a \prec b)$—or, equivalently, if $b \prec a$, there are no parts of b which bear $\approx$ to a.

Suppose otherwise, for reductio. That is, suppose $b \prec a$ but b has a part, $a' \approx a$. By (Properly Extensive), since a' is a voluminous part of b, there must exist some other voluminous part $c \approx c$ such that $a' \circ c = b$. However, by (Properly Extensive) and $b \prec a$, a must have a part $b' \approx b$ and another part, $d \approx d$, such that $b' \circ d = a$.

But, by (Properly Extensive) and (Additivity), b' must partition into non-overlapping parts $a'' \approx a$ and $c' \approx c$ such that $a'' \circ c' = b'$. Since c' and d (both parts of a) are voluminous and don't overlap, it follows, by (V-Comb), that their mereological fusion, $(c' \oplus d)$, is voluminous. However, this means that a is the fusion of two non-overlapping voluminous parts, a'' and $(c' \oplus d)$, i.e. $a'' \circ (c' \oplus d) = a$. But the Within-Object Archimedean Assumption implies that there are no zero magnitudes of volume (see page 18 in the main text), i.e. that there can be no $xy \circ z$ such that $y \approx y$ and $x \approx z$. But, this means that, given $a'' \circ (c' \oplus d) = a$, it must be that $a \not\approx a''$. Contradiction! So b does not have a voluminous part $\approx a$. $\square$

We'll call a copy of x something with the same volume as x. What Lemma 15 says is that the fusion of n-many non-overlapping copies of x has the same volume as the fusion of m-many non-overlapping copies of x just in case $n = m$. If we wanted to sound like Euclid, we'd say something like this: According to Lemma 15, equal numbers of equal volumes are equal.

Lemma 15. *Let S be a set of n objects, all of which bear $\approx$ to some a, none of which overlap. Let S^* be another such set with the same cardinality. $fus(S) \approx fus(S^*)$ (where '$fus(X)$' denotes the fusion of all the members of the set, X).*[57]

Proof. If $n = 1$, then S and S^* are each singletons whose sole member is $\approx a$. $fus(S) \approx a$ and $fus(S^*) \approx a$. By ($\approx$ Trans), $fus(S) \approx fus(S^*)$.

Now we show that, if $fus(S) \approx fus(S^*)$ for some $n \in \mathbb{Z}$, then $fus(S) \approx fus(S^*)$ for $n+1$.

Let $S = \{x_1, x_2, ..., x_{n+1}\}$ and $S^* = \{y_1, y_2, ..., y_{n+1}\}$. Since $S \setminus \{x_{n+1}\}$ and $S^* \setminus \{y_{n+1}\}$ are sets of n-many non-overlapping objects all $\approx a$, by our assumption: $fus(S \setminus \{x_{n+1}\}) \approx fus(S^* \setminus \{y_{n+1}\})$.

[56] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[57] The proof of this lemma directly depends on Lemma 13. For the full map of Lemma interdependence, see figure 4, p. 59.

Since $x_{n+1} \approx a$ and $y_{n+1} \approx a$, $x_{n+1} \approx y_{n+1}$ by ($\approx$ Trans). Note that $fus(S)$ is just the fusion of $fus(S \setminus \{x_{n+1}\})$ and x_{n+1}, i.e. $(fus(S \setminus \{x_{n+1}\}), x_{n+1}) \circ (fus(S))$. Similarly $(fus(S^* \setminus \{y_{n+1}\}), y_{n+1}) \circ (fus(S^*))$. By Lemma 13, since $fus(S \setminus \{x_{n+1}\}) \approx fus(S^* \setminus \{y_{n+1}\})$ and $x_{n+1} \approx y_{n+1}$, it follows that $fus(S) \approx fus(S^*)$.

So if $fus(S) \approx fus(S^*)$ for some $n \in \mathbb{Z}$, then $fus(S) \approx fus(S^*)$ for $n+1$. Hence, since $fus(S) \approx fus(S^*)$ for $n = 1$, it follows that $fus(S) \approx fus(S^*)$ for all cardinalities, $n \in \mathbb{Z}^+$. □

B.4 Proper Extensiveness

This section proves that the mereological expression of proper extensiveness in 3.6 entails the second-order version of these conditions in my "Properly Extensive Quantities", and vice versa (given some reasonable assumptions).

The expression of proper extensiveness I presented in Perry (2015) was second-order in that it concerned quantitative ordering and summation relations between the determinate magnitudes of a given properly extensive quantity. Here are the necessary conditionals, with notation modified for clarity and to fit with the rest of this chapter, that I took to characterize proper extensiveness:

(Additive 'Less') $$\textsc{Less}(V_m, V_n) \rightarrow \forall x \forall y((V_n(x) \wedge V_m(y)) \rightarrow \neg Pxy)$$

(Additive 'Sum') $$\textsc{Sum}(V_m, V_n, V_r) \rightarrow \forall x \forall y \forall z((V_m(x) \wedge xy \circ z) \rightarrow (V_r(z) \leftrightarrow V_n(y)))$$

(Extensive 'Less') $$\textsc{Less}(V_m, V_n) \rightarrow \forall x(V_n(x) \rightarrow \exists y(y \neq x \wedge V_m(y) \wedge Pyx))$$

(Extensive 'Sum') $$\textsc{Sum}(V_m, V_n, V_r) \rightarrow \forall x(V_r(x) \leftrightarrow \exists y \exists z(V_m(y) \wedge V_n(z) \wedge yz \circ x))$$

There's a debate to be had about exactly how to define the second-order relations, 'Less' and 'Sum', between determinate volume properties (V_i's) in terms of the first order relations between voluminous objects. It's likely that the right answer would involve some modal notions (e.g., saying that, necessarily, a given property is instantiated by some object *o* if and only if *o* has parts with such-and-such other volume properties). I won't weigh in on this here.

Rather, I'll rely on the assumption of some simple bridge laws, saying that the first-order summation and ordering relations should obtain between objects just in case the corresponding second-order relations hold between their respective volume. That is, I will assume the biconditionals "*c* is as voluminous as *a* and *b* put together iff their respective volumes stand in the $\textsc{Sum}(x, y, z)$ relation" and "*a* is less voluminous than *b* just in case *a*'s volume bears $\textsc{Less}(x, y)$ to *b*'s volume". It's a necessary condition of any adequate second-order theory of quantity, regardless of how or whether it defines the second-order quantitative relations, that it satisfy (for all volume magnitudes V_m, V_n, V_r), (Bridge 1) $\forall x, y((V_m(x) \land V_n(y)) \to (x \prec y \leftrightarrow \textsc{Less}(V_m, V_n)))$

and

(Bridge 2)
$\forall x, y, z((V_m(x) \land V_n(y) \land V_r(z)) \to (z \text{ is as voluminous as } x \text{ and } y \text{ put together} \leftrightarrow \textsc{Sum}(V_m, V_n, V_r)))$

While the M-R account is technically a property-theoretic account, its only appeals to properties concern whether or not a and b instantiate the *same* volume magnitude ($a \approx b$). As such, it is ill-equipped to deal with cases of uninstantiated volume properties. In what follows, I will assume that any volume magnitude which appears in a logically atomic sentence (e.g. instances of '$\textsc{Less}(x, y)$' or '$\textsc{Sum}(x, y, z)$') is instantiated by at least one object.[58]

Given these assumptions and our two bridge laws, I will show (Lemmas 16–19) that the conditionals I introduce in Perry (2015) follow from the M-R account of volume, specifically (Additivity) and (Properly Extensive). I'll then show that (Additivity) and (Properly Extensive) can be derived from those four conditionals along with some reasonable assumptions (Lemmas 20–21).[59]

Lemma 16. , **(Additive 'Less')** *If* $\textsc{Less}(V_m, V_n)$ *then* $\forall x \forall y((V_n(x) \land V_m(y)) \to \neg Pxy)$

Proof. Suppose that $\textsc{Less}(V_a, V_b)$. Consider an arbitrary pair a and b such that $V_a(a)$ and $V_b(b)$. By (Bridge 1), this implies that $a \prec b$.

By Lemma 14, if $a \prec b$, there are no parts of a which bear $\approx$ to b. Since $b \approx b$, this implies that $\neg P(ba)$. So $\neg P(ba)$ follows from the assumption that $V_a(a)$ and $V_b(b)$. Since this was shown for an arbitrary pair, a and b, it's true for all such pairs, i.e. $\forall x \forall y((V_b(x) \land V_a(y)) \to \neg Pxy)$.

□

Lemma 17. , **(Additive 'Sum')** *If* $\textsc{Sum}(V_m, V_n, V_r)$ *then* $\forall x \forall y \forall z(V_m(x) \land xy \circ z \to (V_r(z) \leftrightarrow V_n(y)))$

Proof. Suppose that $\textsc{Sum}(V_a, V_b, V_c)$. By the instantiation assumption, $V_a(a)$, $V_b(b)$, and $V_c(c)$ for some a, b, and c. By (Bridge 2), c is as voluminous as a and b put together. Or, equivalently, there exist some $a' \approx a$ and $b' \approx b$ such that $a'b' \circ c$.

Now consider an arbitrary trio: s, t, u, such that $V_a(s)$ (i.e. $s \approx a$) and $st \circ u$. It suffices to show that $V_c(u) \leftrightarrow V_b(t)$ (equivalently, $u \approx c \leftrightarrow t \approx b$).

[58] A discussion which tackles the issues with uninstantiated magnitudes head-on would require a detailed account of the modal profile of these properties, specifically under what conditions it's necessary that they be instantiated or not instantiated, and so on. I've already said that getting into that debate would take us too far afield.

[59] The proofs for lemmas 16 and 18 directly depend on Lemma 14, while the proofs for lemmas 17 and 19 depend on Lemma 13. For the full map of Lemma interdependence, see figure 4, p. 59.

> *Left to right:* Suppose that $V_b(t)$, i.e. that $t \approx b$. Since $s \approx a$, $t \approx b$, $st \circ u$ and $ab \circ c$, it follows from Lemma 13 (Unique Sum Lemma), that $u \approx c$. Since $V_c(c)$, this implies that $V_c(u)$.
>
> *Right to left:* Suppose that $V_c(u)$, i.e. that $u \approx c$. By (Additivity) and $ab \circ c$: if $u \approx c$ and $st \circ u$, then $s \approx a$ just in case $t \approx b$. Since we've supposed that $V_a(s)$, i.e. that $s \approx a$, it follows that $t \approx b$. Since $V_b(b)$, this implies that $V_b(t)$.

So $V_c(u) \leftrightarrow V_b(t)$ follows from the assumption that $V_a(s)$ and $st \circ u$. Since this was shown for an arbitrary trio s, t, and u, this conditional holds universally. □

Lemma 18. , **(Extensive 'Less')** *If* Less(V_m, V_n) *then* $\forall x(V_n(x) \rightarrow \exists y(y \neq x \wedge V_m(y) \wedge Pyx))$

Proof. Suppose that Less(V_a, V_b). Consider an arbitrary b such that $V_b(b)$. By the instantiation assumption, there exists some a such that $V_a(a)$. It suffices to show that there exists some part, c, of b such that $c \approx a$ and $c \neq b$.

By (Bridge 1), $a \prec b$. By the definition of $\prec$, there exists a part, c, of b such that $c \approx a$. Since $V_a(c)$ and $V_a \neq V_b$, $c \not\approx b$ so $c \prec b$. By Lemma 14, if $c \prec b$, then there are no parts of c which bear $\approx$ to b. If c were identical to b, then c would have a part that bears $\approx$ to b, hence $c \neq b$. Since this was shown for an arbitrary b such that $V_b(b)$, it's true in general, i.e. $\forall x(V_b(x) \rightarrow \exists y(y \neq x \wedge V_a(y) \wedge Pyx))$. □

Lemma 19. , **(Extensive 'Sum')** *If* Sum(V_m, V_n, V_r) *then* $\forall x(V_r(x) \leftrightarrow \exists y \exists z(V_m(y) \wedge V_n(z) \wedge yz \circ x))$

Proof. Suppose that Sum(V_a, V_b, V_c).

> *Left to Right:* Pick some arbitrary object c such that $V_c(c)$. By our original supposition plus the instantiation assumption, there exists some a and b such that $V_a(a)$ and $V_b(b)$. By (Bridge 2), c is as voluminous as a and b put together. Or, equivalently, there exist some $a' \approx a$ and $b' \approx b$ such that $a'b' \circ c$. So the left to right half of the biconditional is established.
>
> *Right to left:* Let d be an arbitrary entity such that there exists some x and y such that $V_a(x)$, $V_b(y)$, and $xy \circ d$. By our original supposition plus the instantiation assumption, there exists some c such that $V_c(c)$. By (Bridge 2), c is as voluminous as x and y put together, i.e. there exists some $x' \approx x$ and $y' \approx y$ such that $xy \circ c$. By Lemma 13 (Unique Sum) if $xy \circ d$ and $x'y' \circ c$, where $x' \approx x$ and $y' \approx y$, it follows that $d \approx c$. Since $V_c(c)$, this means that $V_c(d)$. So the right to left half of the biconditional is established.

Since both directions of the biconditional are established given an arbitrary individual satisfying each antecedent, the biconditional holds universally. That is, $\forall x(V_c(x) \leftrightarrow \exists y \exists z(V_a(y) \wedge V_b(z) \wedge yz \circ x))$. □

The next step will be to prove that the axioms which characterize volume's proper extensiveness on the M-R account follow from my original formulation, given the bridge laws and some reasonable assumptions (which I'll go into during the proof of Lemma 21, below). I do not presuppose the axioms of the M-R account of volume, with the exception of the volume combination principle, (refvcomb), which says that if a pair of voluminous entities are put together in the right way, their fusion is voluminous.

Lemma 20. *The axiom, (Additivity)*

$$\text{(Additivity)} \qquad a \approx a \wedge b \approx b \wedge ab \circ c \;\rightarrow \; \forall x \forall y \forall z \Big(x \approx c \wedge yz \circ x \rightarrow \; (y \approx a \rightarrow z \approx b) \Big)$$

follows from (Additive 'Less'), (Additive 'Sum'), (Extensive 'Less'), *and* (Extensive 'Sum').[60]

Proof. Suppose there exist some V_a, V_b such that $V_a(a)$ and $V_b(b)$ and $ab \circ c$. By (V-Comb) $\exists V_c$ such that $V_c(c)$. Therefore, by definition, c is as voluminous as a and b put together. By (Bridge 2) $\textsc{Sum}(V_a, V_b, V_c)$.

We want to show that, for any x, y, z such that $V_c(x)$ and $yz \circ x$, $V_a(y) \rightarrow V_b(z)$. To show this, suppose x is such that $V_c(x)$ and $yz \circ x$. Further, suppose that $y \approx a$ (i.e. $V_a(y)$). Suffices to show that $z \approx b$.

Since $\textsc{Sum}(V_a, V_b, V_c)$, by (Additive 'Sum') $yz \circ x$ and $V_a(y)$ imply that $V_c(x) \leftrightarrow V_b(z)$. $V_c(c)$, and we've assumed that $V_c(x)$, so $x \approx c$. So it it follows that $V_b(z)$ and, since $V_b(b)$, $z \approx b$. □

Lemma 21. *The axiom, (Properly Extensive)*

$$a \approx a \wedge Pab \wedge b \approx d \rightarrow (a \approx b \vee \exists x \exists y (x \approx a \wedge y \approx y \wedge xy \circ d))$$

follows from (Additive 'Less'), (Additive 'Sum'), (Extensive 'Less'), *and* (Extensive 'Sum'), *plus the assumption that* $\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r))$ *for all* V_m, V_n, *and* V_r.[61]

Proof. Let a, b, and d be arbitrary individuals such that a is a part of b and there exist some volume properties, V_a, V_b, such that $V_a(a)$, $V_b(b)$, and $V_b(d)$.

We want to show that either $a \approx b$ or $\exists x \exists y \exists V_y (x \approx a \wedge V_y(y) \wedge xy \circ d)$. Either $V_a = V_b$ or not. We reason by cases: If $V_a = V_b$, then $a \approx b$ by the definition of $\approx$. Suppose, instead, that $V_a \neq V_b$.

If $V_a \neq V_b$, then $a \not\approx b$. By the definition of '$\prec$', since Pab and $a \not\approx b$, $a \prec b$. By (Bridge 1), this means that $\textsc{Less}(V_a, V_b)$. Since $V_b(d)$, (Bridge 1) also implies that $a \prec d$. By the definition of '$\prec$', there's a part, a', of d such that $a' \approx a$.

[60] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

[61] The proof of this lemma depends on no other lemmas. For the full map of Lemma interdependence, see figure 4, p. 59.

This establishes that d has a proper part $a' \approx a$. However, we need an additional assumption to get to (Properly Extensive). Specifically, we need to assume that $\forall V_m, V_n(\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r)))$. This makes sense. The conditionals presented in Perry (2015) were never meant to stand alone. Rather, they serve to connect a Mundy-style second-order account of quantitative structure to the first-order mereological structure. On a Mundy-style account of volume, primitive axioms govern the distribution of the second-order summation and ordering relations over the first-order volume properties. The analogue of "$\forall V_m, V_n(\textsc{Less}(V_m, V_n) \rightarrow \exists V_r(\textsc{Sum}(V_m, V_n, V_r)))$" for mass is entailed by the account in Mundy (1987), so it's reasonable to assume it would hold here.

If we accept this conditional, then we can infer, from $V_a(a)$, $V_b(d)$, and $\textsc{Less}(V_a, V_b)$, that there exists some volume property, V_x, such that $\textsc{Sum}(V_a, V_x, V_b)$. The rest follows from (Extensive '$\textsc{Sum}$'). Specifically, $\textsc{Sum}(V_a, V_x, V_b)$ and $V_b(d)$ imply that $\exists y \exists z(V_a(y) \wedge V_x(z) \wedge yz \circ d)$. Hence $\exists y \exists z(y \approx a \wedge z \approx z \wedge yz \circ d)$.

Since it follows from both $V_a = V_b$ and its negation, the disjunction, $a \approx b$ or $\exists x \exists y \exists V_y(x \approx a \wedge V_y(y) \wedge xy \circ d)$, follows from the supposition that a is a voluminous part of b and $b \approx d$. Since the conditional holds for an arbitrary a, b, and d, it holds for all such trios. □

B.5 Continued Fractions

In this section I'll show that every real number can be uniquely expressed as a simple continued fraction. These are not new results, but in case the reader is not familiar with these features of continued fractions, there are certain easy and intuitive ways to make it clear that they have these properties.

Let $n \in \mathbb{R}$ be some real number. First I will show that n can be expressed as a simple continued fraction, then I will show that this expression is unique. This presentation closely follows Chrystal (1886).

Consider the integer a_0, where a_0 is the greatest integer such that $a_0 \leq n$. So

(25)
$$n = a_0 + \frac{1}{n_1}$$

where $n_1 > 1$ is some real number. Now let a_1 be the greatest integer $\leq n_1$. Then

(26)
$$n_1 = a_1 + \frac{1}{n_2}$$

where $n_2 > 1$ as before. Again, let a_2 be the greatest integer $\leq n_2$, then

(27)
$$n_2 = a_2 + \frac{1}{n_3}$$

This process will terminate if some n_i is an integer, for then we'd have

$$n_i = a_i \tag{28}$$

and there would be no n_{i+1}. Also, since n_i is the result of a process like the one above, it must be > 1, and so $a_i > 1$. If there is no such n_i the process will not terminate.

From these we get that

$$n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{29}$$

or

$$n = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots \tag{30}$$

Regardless of whether this process terminates, we have resulted in either a terminating or infinite simple continued fraction expression of n, since $n \in \mathbb{R}$ is an arbitrary real number, we can conclude that all real numbers can be expressed as simple continued fractions.

Now to prove that the continued fraction expression of a real number is unique. It suffices to show that for any n such that

$$n = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots = a_0' + \frac{1}{a_1'+}\,\frac{1}{a_2'+}\,\frac{1}{a_3'+}\cdots \tag{31}$$

Each term is equal, i.e. $a_0 = a_0'$, $a_1 = a_1'$, $a_2 = a_2'$, etc.

We know, from the process by which continued fractions are constructed, that a_0 and a_0' are positive integers, and that $\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots$ and $\frac{1}{a_1'+}\,\frac{1}{a_2'+}\cdots$ are both positive (or zero) and < 1 (since every a_i and a_i' is positive). Suppose (WLOG) that $a_0 < a_0'$. Then $a_0' \geq a_0 + 1$ (since they are both integers).

But then

$$a_0' + \frac{1}{a_1'+}\,\frac{1}{a_2'+}\,\frac{1}{a_3'+}\cdots \geq (a_0 + 1) + \frac{1}{a_1'+}\,\frac{1}{a_2'+}\,\frac{1}{a_3'+}\cdots \tag{32}$$

Since, by (31) the primed and unprimed continued fractions are equal, it follows that

$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots \geq a_0 + 1 + \frac{1}{a_1'+}\,\frac{1}{a_2'+}\,\frac{1}{a_3'+}\cdots \tag{33}$$

Subtracting out a_0, we have

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots \geq 1 + \frac{1}{a'_1+}\,\frac{1}{a'_2+}\,\frac{1}{a'_3+}\cdots \tag{34}$$

But we know that $\frac{1}{a'_1+}\frac{1}{a'_2+}\cdots \geq 0$, hence $\frac{1}{a_1+}\frac{1}{a_2+}\cdots \geq 1$. But we know that $\frac{1}{a_1+}\frac{1}{a_2+}\cdots < 1$. So the assumption that $a_0 < a'_0$ leads to a contradiction, hence $a_0 \geq a'_0$. Parallel reasoning shows that $a'_0 \geq a_0$. Hence $a_0 = a'_0$.

If the first terms are equal, then

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots = \frac{1}{a'_1+}\,\frac{1}{a'_2+}\,\frac{1}{a'_3+}\cdots \tag{35}$$

which means that their inverses[62] are equal, i.e.

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots = a'_1 + \frac{1}{a'_2+}\,\frac{1}{a'_3+}\,\frac{1}{a'_4+}\cdots \tag{36}$$

Here a_1 and a'_1 are both positive integers, and $\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\ldots$ and $\frac{1}{a'_2+}\frac{1}{a'_3+}\frac{1}{a'_4+}\ldots$ are positive (or zero) and < 1. By the same reasoning as before, $a_1 = a'_1$

We can continue this process, and can show that each $a_i = a'_i$, even if the fractions never terminate.[63]

Hence continued fractions can be used to uniquely pick out real numbers. Continued fractions are structured such that they can be expressed as lists of positive integers (i.e. $\langle a_0, a_1, a_2, \ldots\rangle$) which may or may not terminate (and where the final integer if there is one, is > 1). The ratio procedure for a pair of voluminous entities, generates a list of positive integers, which may or may not terminate (and where the final integer, if there is one, is > 1). Hence, the list of integers generated by the ratio procedure for any voluminous pair *uniquely* corresponds to the continued fraction expansion of a positive real number.

[62] What does it mean to take the inverse of a continued fraction, given that these are things which may be infinitely long? The operation is actually not as mysterious as it might seem. We are taking the inverse of a fraction whose numerator is = 1. In this case, the inverse is just the denominator of that fraction.

[63] Technically, one would have to prove that infinite simple continued fractions converge in the right way, so that the reasoning, above, using the continued fractions make sense even when they don't terminate. This is a well known result, and can be seen in (Chrystal, 1886, p. 441) , and (Davenport, 1962, p. 93).

C Map of Dependencies Between Lemmas

Here I include a graphical map of the dependency relations between the 21 Lemmas featured in the text.

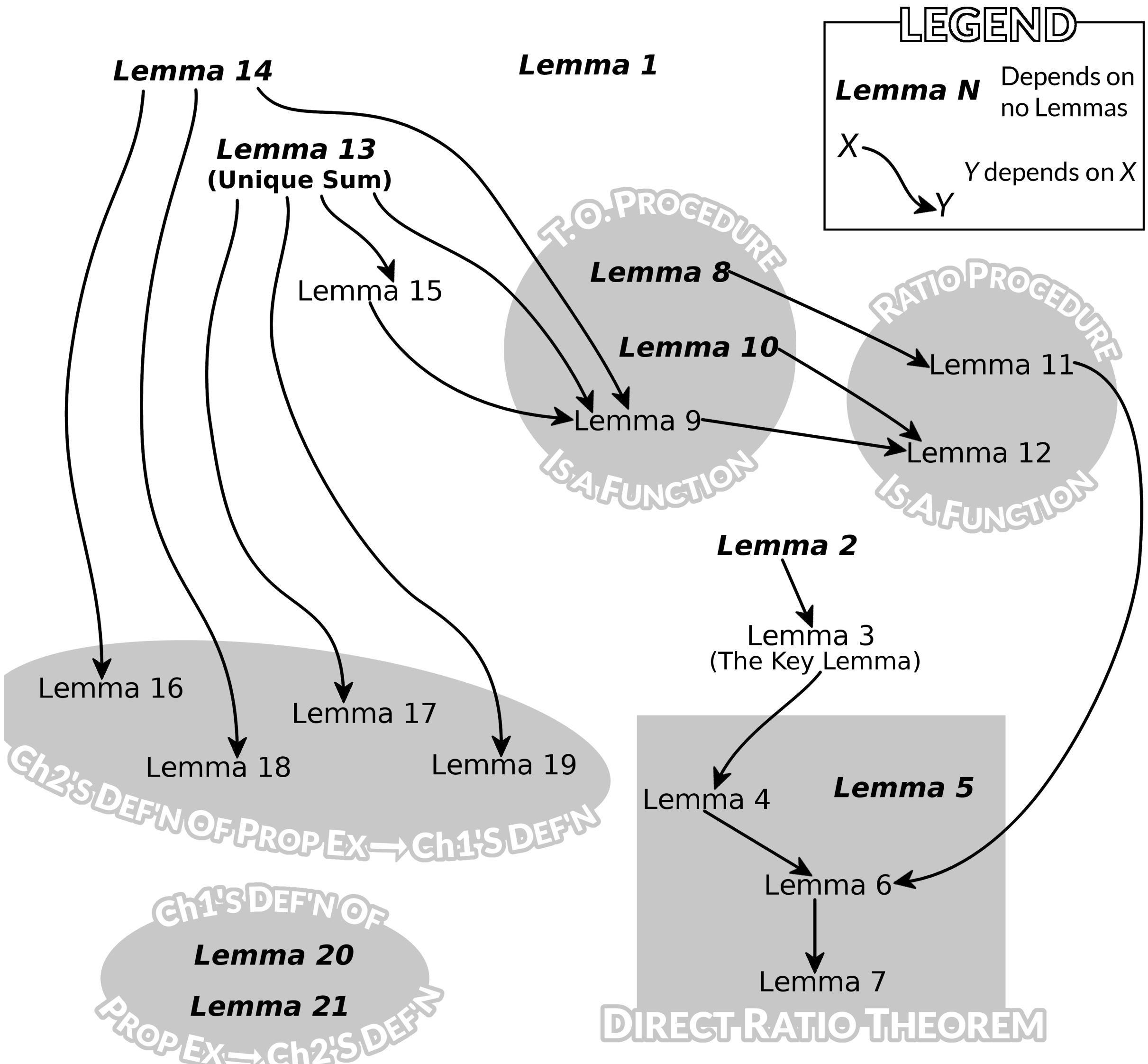

Figure 4: A graphical depiction of the dependency relations between the lemmas proved in the text. Since every dependency arrow follows a downward trajectory, this figure also serves as a visual proof that the dependence of some lemmas on others does not commit us to any circularity.

Lemma 1: p. 20
Lemma 2: p. 32
Lemma 3: p. 33
Lemma 4–7: p. 39–42
Lemma 8–10: p. 44–46
Lemmas 11, 12: p. 48
Lemma 13–15: p. 51
Lemmas 16–19: p. 53–54
Lemma 20, 21: p. 55