Does Physics Motivate a Dynamic Theory of Quantity?

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Abstract: This paper defends and motivates a "dynamic account" of mass, a theory on which mass's quantitative structure (specifically, mass ratio relations) is metaphysically dependent on the structure of *other* quantities (viz. spatiotemporal quantities like length or acceleration) together with the laws of nature. For instance, given $\vec{F} = m\vec{a}$, a dynamic theory of mass grounds the mass ratio "n-times as massive as" in the relative tendency to accelerate at $1/n$-times the rate when under an equally strong force.

This account is philosophically fruitful, in that it solves a difficult puzzle in the metaphysics of quantity concerning the possibility of underpopulated worlds. In addition, considerations from the physics—specifically, how the dynamical laws treat these quantities in different possible worlds—strongly suggest that (1) mass is dependent, for its structure, on length and temporal duration, and (2) this dependence obtains only in virtue of the dynamical laws being what they are.

1 Introduction

We represent physical quantities, in science and our everyday practice, using mathematical entities like numbers and vectors. We use a number together with a unit to denote determinate magnitudes of mass or length (like 2kg, 7.5m etc.), and then appeal to the mathematical relationships between those numbers to describe and explain physical phenomena. These mathematical representations are appropriate insofar as they mirror some structure inherent in the physical world and the systems it describes.

An account of "quantitative structure" is an account of the physical entities, properties, and relations that underlie these explanations – i.e. it is an account of those aspects of the physical world we're latching on to when we use mathematical notions to describe/represent it. In what follows, I articulate two, closely related, considerations in favor of a type of account which grounds mass's quantitative structure in the structure of *other* quantities, in particular spatiotemporal quantities like length and temporal duration. Call this a "hierarchical" account of mass.

I'll argue that the best hierarchical accounts don't take the connection between mass and spatiotemporal quantities as *primitive*, but, instead, appeal the connections between those quantities already present in the physics. A "dynamic theory of quantity" grounds the structure of one quantity in the structure of another via the connection between them imposed by the *dynamical laws*—e.g. the way that $\vec{F} = m\vec{a}$ says that the acceleration (definable in terms of length and duration) of an object under a force is in proportion to its mass.[1]

I'll argue, specifically, that a dynamic account of mass in terms of length and temporal duration is both theoretically fruitful and physically well-motivated. It's theoretically fruitful in that it solves a widespread problem in the metaphysics of quantity, and it's physically well-motivated in that considerations from the dynamical laws suggest that mass's structure is dependent on or determined by the structure of spacetime.

[1] The idea of a dynamic theory of quantity is not new. Furthermore, there's precedence for accounts that specifically appeal to the dynamical connection between quantities mass and quantities like length, volume, and temporal duration. For instance, Russell (1903, Part III, Chap XXI.165) describes something like this view. There's also the account, proposed by Mach (1893), which defines mass in terms of its dynamics.

2 Why a Dynamic Account of Quantity?

In this section, I present a problem for any theory of physical quantity, and describe how it's especially problematic for certain accounts of mass. I then show how one can avoid this problem by adopting a hierarchical account of mass, which grounds mass's structure in the structure of another quantity. Finally, I'll suggest a way we might fit the metaphysical scaffolding of a hierarchical account of quantity onto our physics in a non-arbitrary way by appealing to the *dynamical laws* to "link up" mass ratios with ratios of length/temporal duration (or with ratios of quantities definable in terms of them).

2.1 Measurement-Theoretic Accounts of Quantity

Let's lay down some desiderata. We want a theory of quantity that can define ratio relations, like "2.7-times as massive as", in a way that matches up with our mathematical representations. Moreover, we want this definition to be *explanatory*, in that the question "What does it mean for A to be 2.7-times as massive as B?" should have a clear and non-trivial answer. And, importantly, we want that answer to make contact with the physics; Our understanding of quantitative structure should, that is, ultimately be about the physical world and the role of those quantities in it.

Each mass ratio relation is, from a physical perspective, a distinct physical relation with substantially different consequences for the behavior of its relata. There's no prospect for defining each of them individually, but there's a widely-used method for simplifying this task by reducing facts about mass *ratios* to facts about the world satisfying certain measurement-theoretic axioms.[2] Measurement theory is a formal discipline which involves rationalization, defense, and formal reconstruction of our empirical measurement practices. The game of measurement theory is as follows: First, take a domain of, e.g.,

[2]Field (1980) is the most famous philosophical account along these lines.

massive objects, together with the distribution of certain non-ratio relations over them, like a mass ordering or mass summation. Then, posit some axioms that these relations obey and prove "representation and uniqueness theorems" which imply that that domain can be faithfully represented (up to a point) with a certain mathematical structure—specifically one with rich ratio structure, like the real numbers.[3]

This is a very popular solution to the problem, but it's viability rests entirely on its ability to prove these theorems given the actual ordering and summation facts. This is an issue, since certain axioms needed to prove these theorems rely on the domain being *well-populated* (existence axiom), and that there's ample *variation in the mass properties* instantiated therein (richness axiom). Whether these axioms are satisfied is a contingent matter, which leads to the problem of underpopulation.

The Problem of Underpopulation

Call an "impoverished mass world" one where the actual massive bodies fail to satisfy the existence and richness assumptions needed to prove the right representation and uniqueness theorems, i.e. where the actual distribution of mass ordering and summation relations fail to uniquely determine the mass *ratio* relations. Some physical theories—for instance, Newtonian mechanics with massive point particles—are rife with such possibilities. Two massive particles may stand in the "π-times as massive as", or the "twice as massive as" or any other of countless ratio relations, even if they're the only massive bodies in the entire world.

Impoverished mass worlds pose a serious problem for any account of quantity aiming to recapture the predictions of our physical theories. Consider a world containing particles A, B and C (Figure 1), in a two-dimensional spacetime, with A and B on course to

[3] Cf. Krantz et al. (1971)

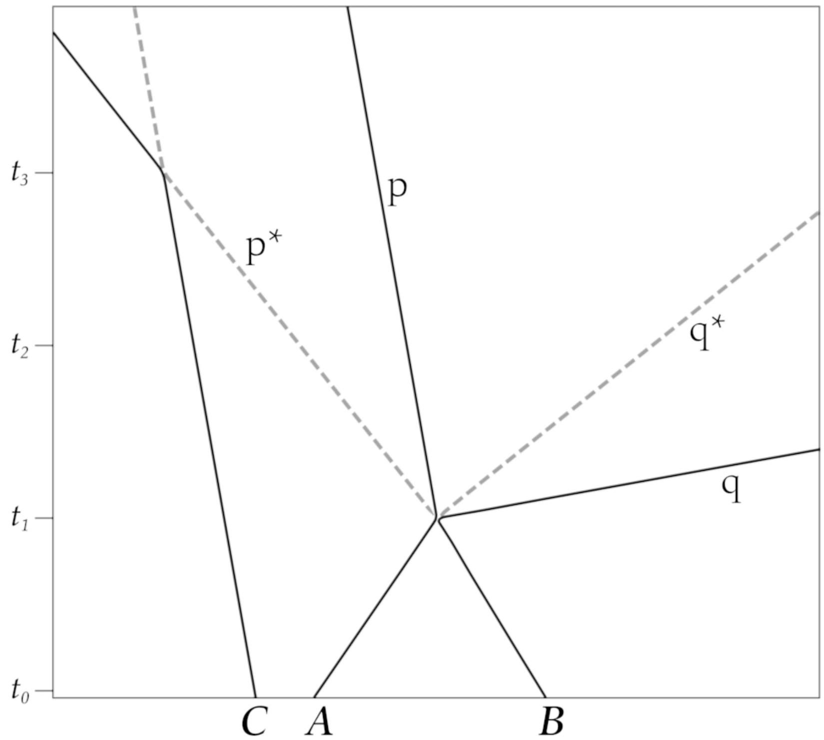

Figure 1: Two possible evolutions of a massive trio.

collide at t_1. What happens after t_1 depends on the mass ratio between A and B. If A is exactly 3π-times the mass of B, then A will be deflected along trajectory, p, which never intersects C's trajectory. If, however, A is *less than* 3π-times the mass of B, A will instead follow a trajectory like p^*, resulting in a later collision with C. However, the mass ordering and summation relations between A, B, and C (the domain of massive objects at this world), will almost always fail to uniquely determine their mass ratios.[4]

Various theorists have proposed ways to avoid this problem. Some just add the as-

[4]Excepting in cases where some of them are equally massive

sumption that the world is always well-populated and variegated enough to prove the right representation and uniqueness theorems. For instance, Field (1980) rejects point particles, requiring instead that mass density vary continuously (meaning that, for any two regions that differ in mass density, there will be regions possessing *each* of the mass density properties in between them). Others, like Mundy (1987), change the domain. Mundy applies the same measurement theoretic to the domain of (necessarily existing) mass *properties*, rather than particular massive things. Similarly, Arntzenius and Dorr (2012) posit a substantival "mass space", each point of which is a determinate mass property.

2.2 Hierarchical Accounts of Quantity Solve the Problem

A hierarchical account of mass responds to the problem of underpopulation by appealing to *other* quantities, specifically ones whose *actual* structure is richer than mass's. That way, the mass ratios are not limited by the actual mass ordering and summation facts. This doesn't completely eliminate the problem, since we must account for the ratio structure of *those* quantities, but it reduces two instances of this problem into one.

Here's a simple hierarchical account: Let $Q_{\text{L}\to\text{M}}$ be a primitive four-place relation such that, intuitively, "$a, b\, Q_{\text{L}\to\text{M}}\, l_1, l_2$" just in case the *mass ratio* between a and b is "the same as" the *length ratio* between the two *spatial paths*, l_1 and l_2. Call this a "mixed relation" because it links the structure of one quantity with the structure of another. If we posit some simple axioms for $Q_{\text{L}\to\text{M}}$,[5] this will allow us to define mass ratios using the ratio structure of spatial length.[6]

Such theories are of great formal and metaphysical convenience. So long as there's

[5]For instance: "$a, b\, Q_{\text{L}\to\text{M}}\, l_1, l_2$ iff $b, a\, Q_{\text{L}\to\text{M}}\, l_2, l_1$",
"If $a, b\, Q_{\text{L}\to\text{M}}\, l_1, l_2$ and $b, c\, Q_{\text{L}\to\text{M}}\, l_2, l_3$, then $a, c\, Q_{\text{L}\to\text{M}}\, l_1, l_3$",
"If $a, b\, Q_{\text{L}\to\text{M}}\, l_1, l_2$, then $a, b\, Q_{\text{L}\to\text{M}}\, l_3, l_4$ iff l_3 and l_4 stand in the same length ratio as l_1 and l_2", etc.

[6]Cf. Burgess (1984) and (1991) for an account of mass roughly along these lines.

enough material for the theory to ground *length* ratios, then the mass ratios are also covered, no matter how few massive things actually exist. This is useful, as the threat of impoverished worlds is not the same for all quantities. It's decidedly less controversial, that is, to say that the domain of *lengthy* entities is necessarily well-populated and richly variegated. Indeed, this is a direct consequence of substantivalism about space and time![7]

2.3 Motivating a Hierarchical Account of Quantity with Dynamics

So, even at worlds with only a few massive objects, substantivalism about space ensures that there will be a rich enough actual length ratio structure to ground, by way of a relation like $Q_{\mathrm{L}\rightarrow\mathrm{M}}$, the mass ratios relations at that world.

Recall that I said our metaphysics of quantity should take its marching orders from the physics and the role of those quantities in the physical world. This is where "brute" hierarchical accounts run into trouble. Physics doesn't make use of cross-quantity "same ratio as" relations. What motivation is there, then, to not only take $Q_{\mathrm{L}\rightarrow\mathrm{M}}$ as *fundamental*, but also to use that relation to ground facts that *do* appear in the physics (like mass ratios)?

Luckily, we can do better. Some parts of the physics, like the dynamical laws, *do* privilege a connection between ratios of different quantities, without need for a fundamental mixed relation.

Here's how the dynamics could privilege a pairing of mass ratios with ratios of another quantity: Laws like $\vec{F} = m\vec{a}$ associate a body's mass with its *acceleration*. If particles, a and b, are impressed by the same force, then their relative accelerations will be determined by their mass ratio. So, if a is twice as massive as b, then b will accelerate twice as much as a in response to that force. One could, then, ground mass ratios ("n-times as massive as") in

[7] Field (1984) presents this result as an argument for substantivalism. See also Meinong (1896) and, more recently, **?**, who suggest it's an inherent feature of quantities like length that their instances be divisible into less lengthy parts.

terms of law-governed tendency to accelerate $1/n$-times as much under an equally-strong force.

A dynamic account of mass (like any hierarchical account) is unbothered by impoverished mass worlds. Even when there are too few massive objects to prove a representation theorem, we appeal to the connections between mass and acceleration imposed by the dynamical laws to determine the mass ratios using acceleration ratios. Since acceleration and velocity are defined quantities mass ratios are, ultimately, grounded in the ratio structure(s) of spatiotemporal length (used as an umbrella term to incorporate both spatial length and temporal duration).

2.4 Two Remaining Problems

How much of an improvement is a dynamic account over a brute hierarchical account?

One might worry that merely "making contact" with the physics is not enough. The pairing between mass ratios and accelerations is certainly less arbitrary than the pairing imposed by a primitive $Q_{\text{L}\to\text{M}}$ relation, but that doesn't mean that there's any physical support for *grounding* mass ratios in acceleration ratios. Indeed, a dynamic account of quantity assigns a *radically* different metaphysical status to quantities appearing in the same laws—some, like length, have their structure fundamentally, while others, like mass, have their structure partly in virtue of *those very laws*—when no such distinction appears in how the laws *themselves* treat those quantities. If anything, the symmetric treatment of quantities like mass and spatiotemporal length in the physical dynamics tells *against* a dynamic account of quantity.

Moreover, there's an added risk to making metaphysical hay out of something not reflected in the physics. We got here because we wanted a modally sturdy account of mass ratios. But it remains to be shown that spatial length and temporal duration will *always*, as a matter of necessity, be sufficiently rich to determine the mass ratios. Substantivalism about spacetime rules out *some* of the impoverished length worlds, but it's still an open question exactly how much structure spacetime has (space could, for all we know, be discrete). What guarantee is there that mass's ratio structure won't outstrip what its putative spatial grounds can provide?

The next section, I show how both of these worries can be resolved.

3 Does Physics Motivate a Dynamic Theory of Quantity?

I contend that the dynamical laws, at least in certain cases, *do* support some kind of dependence (or asymmetric determination) between mass and spatiotemporal length. This answers the objection from the symmetric treatment of quantities appearing in the laws by denying its main premise.

The argument proceeds by considering worlds that limit the structure of either mass or spatiotemporal length, and determining what kinds of physical facts would be consistent with/permitted by the dynamical laws. While we may not be able to avoid the *metaphysical* possibility of worlds where mass's structure outstrips the ratio structure of spatiotemporal length, we *will* be able to set such worlds aside as *physical/nomological* impossibilities.

3.1 Physical Preliminaries

The following argument uses the theory of Newtonian elastic particle collisions. The only relevant bits of physics will be that (1) given $\vec{F} = m\vec{a}$, a body accelerates in proportion to the force it's under, and (2) from Newton's third law, two colliding bodies feel a force of equal strength, in opposite directions.

I'll consider three worlds. The first is the ordinary Newtonian world, where massive point particles move around in "Neo-Newtonian" or "Galilean" spacetime[8] in accordance with the laws. Mass, length, and temporal duration are all continuous quantities at these worlds.

The second and third worlds are the results of restricting the quantitative facts in different ways. A Discrete *Space* Newtonian world (DSN world) is one where spatial length and temporal duration are both *discrete* quantities, meaning they're better represented by the non-negative integers than the reals, and each simultaneity slice resembles an infinite 3-D lattice. In contrast, a Discrete *Mass* Newtonian world (DMN world) is one where mass is discrete. Otherwise, the DSN and DMN worlds are as similar to ordinary Newtonian worlds as the restrictions mentioned above will allow.

In each world, I'll consider a simple case of particle collision. Massive particles, A and B, approach each other until, at t_1, they collide. My argument will be this: restricting spatiotemporal length's quantitative structure puts strong constraints on the mass ratios A and B can stand in without risk of violating the dynamical laws after t_1. However, this doesn't hold true in the other direction. Restricting the quantitative structure of mass puts no significant constraint on the possible relative velocities A and B may have before t_1, nor on their nomologically possible rates of acceleration immediately after t_1. The differences between how these worlds treat A and B strongly suggests that (1) mass is dependent, for its structure, on length and temporal duration, but (2) this dependence obtains only in virtue of the dynamical laws being what they are.

[8] Wherein each simultaneity slice is a 3-D Euclidean space, and trajectories of bodies over time can be distinguished as straight (inertial motion) or curved (accelerated motion), but there are no physically privileged "rest" trajectories

3.2 Massive Particles in a Discrete Spacetime

Let's say a bit more about spacetime in a DSN world. Each simultaneity slice (corresponding to the whole world at one time) has the structure of a three-dimensional lattice (suppose it's a square lattice), and these simultaneity slices are arranged in a discrete temporal order, with each moment followed by a unique successor moment. This doesn't mean, however, that the *total* spacetime at a DSN world has the structure of a 4-D lattice. Since you can get from any point in space at t_1 to any point in space at the next moment, t_2, there must exist a "minimal trajectory" with those two points as its initial and final endpoints, respectively.

We'll ignore all spatial dimensions other than the 1-D spatial line along which A and B travel before and after t_1. In the smallest span of time (i.e. between one moment and its immediate temporal successor), an object moving along that line could, at best, move a whole number of "steps" in one direction.[9]

Now that we understand what motion in this space amounts to, and how it constrains the possible relative velocities and accelerations of A and B, we can turn to the question at hand: Are there any mass ratios A and B can stand in at an ordinary Newtonian world but which would *not* be nomologically possible at a DSN world? Nothing in the setup of this case put any *metaphysical* restrictions on mass's structure. However, it looks like the answer, once we consider the dynamics, is yes.

The forces imposed on each particle, as a result of their collision, will be equal in strength but opposite in direction. From $\vec{F} = m\vec{a}$, if A and B are under a force of equal strength, the difference in how much they accelerate (acceleration ratio) is the inverse of their mass ratio. That is, if A is twice as massive as B, then it will accelerate *half* as

[9]This way of talking assumes absolute rest. We could drop this assumption at the cost of complicating our explanation. We could talk, instead, about how much a particle has *accelerated* between t_1 and t_3 by considering the distance (at t_3) between the particle's *actual* location and the location it would have ended up at had it continued moving inertially at t_2.

much as B does after colliding. For simplicity, we'll treat both particles' accelerations as instantaneous velocity boosts. So, if B experiences a boost of $24m/s$ in the direction $\vec{d}$, then A must experience a boost of $12m/s$ in the direction $-\vec{d}$.

Now suppose A is π-times as massive as B. The velocity boosts they experience after colliding must stand in the *inverse* of their mass ratio. Velocity ratios are reflected in the ratio of the distances traversed over equal time. But, between each moment and its successor, a particle can only move a whole number of "steps" in any direction, meaning they cannot travel distances that reflect the length ratio 1-to-π. Put another way, there are no velocity pairs in the DSN world such that, possibly, A experiences a boost of $\vec{v}_1$, B experiences a boost of $\vec{v}_2$, and $\vec{v}_2 = -(\pi \cdot \vec{v}_1)$. After they collide at t_1, no future trajectories for A and B are permitted by the Newtonian dynamical laws.

This isn't to say that it's *impossible* for mass to be a continuous quantity at a DSN world. Mass *could* be continuous at a DSN world, but only if either (1) the only mass properties *actually instantiated* by bodies at that world occupy a "discrete substructure" of the continuous class of determinate mass properties,[10] and/or (2) that world doesn't grant the Newtonian dynamical laws a modal profile befitting a law of nature. Regarding the latter, I'm thinking, e.g., of a world where the mass ratio between some pair of bodies, C and D, *would* lead to, at some point, violating $\vec{F} = m\vec{a}$ if C and D were to ever interact (or be subject to similar forces, etc.), but they just so happen to never interact at that world.

So it seems that if we want the dynamical laws to hold true with some degree of necessity, then a restriction on the structure of spatial length and temporal duration requires a similar restriction (either genuine or *de facto*) on the structure of *mass*.

[10]Which we'll define as a maximal substructure, S, such that (1) all actually instantiated mass properties are elements of S; (2) the instantiation, by some body, of any of the actually uninstantiated mass properties in S would not risk conflicting with $\vec{F} = m\vec{a}$; and (3) S is homomorphic to the non-negative integers.

3.3 Discrete Masses in a Continuous Spacetime

Does the same hold for a DMN world? Do the dynamics at a world like this put constraints on what structures of length and temporal duration are possible? A DMN world, recall, has the same spacetime as ordinary Newtonian mechanics. But, at these worlds, mass is a discrete quantity, with each mass property having a unique immediate successor and (except in the case of the minimal mass property) a unique immediate predecessor.

Does a DMN world, given mass's structure and the dynamical laws, put restrictions on the possible relative velocities and accelerations *A* and *B* might possess? The answer in *this* case seems to be no. In a DMN world, bodies and the laws can make full use of Euclidean space's rich quantitative structure despite mass being a discrete quantity.

Here's what I mean: A DMN world does not prevent *A* from moving at π-times the velocity of *B* (*A* and *B* could, for example, start out that way). It also poses no obstacle to *B* *accelerating* from moving 1m/s towards some inertial reference object, *C*, to moving 2πm/s away from it.[11] Hence, there are perfectly acceptable models of the Newtonian laws in a DMN world where the motions of *A* and *B*, before and after t_1, can *only* be captured with a ratio structure as rich as the real numbers (and, therefore, not by the sort of ratio relations you would get if length and temporal duration were discrete). There is no requirement, that is, that *A* and *B*'s trajectories be mappable onto a "discrete subspace" of DMN's spacetime to avoid risk of violating the dynamical laws.

[11] Suppose *A* and *B* are equal in mass, and *A* is moving 2πm/s away from *C* before t_1. Upon colliding, *A* and *B* will "trade" velocities.

4 Conclusions

So there's an asymmetry in how physical quantities interact with the dynamical laws. The structure of mass seems to dependent on (or, at least, partially determined by) the quantitative structure of length and temporal duration—together with the dynamics—and there is no dependence of this sort pointing the other way. A theory of quantity on which mass's quantitative structure is grounded in its dynamic connections to length and temporal duration (whose structure is grounded independently) would provide an elegant explanation for this asymmetry.

In contrast, a hierarchical theory which postulates a fundamental mixed relation, like $Q_{\text{L}\to\text{M}}$, between mass and length must treat this asymmetry as a lucky coincidence. Such a theory rules out certain DSN worlds as *metaphysically impossible*, but it has *no* explanation for why the dynamical laws (which are completely insensitive to the $Q_{\text{L}\to\text{M}}$ relation), just so happen to break down in *precisely those* DSN worlds.

I have argued that dynamic accounts of quantity are theoretically fruitful, in that they solve (or diminish) a major problem in the metaphysics of quantity, and are supported by considerations from (at least some) physics. If we wish to take these arguments seriously, then we have reason to think that mass's quantitative structure is partly dependent on the structure of spatial length and temporal duration, via the laws of nature. A view like this raises questions outside the metaphysics of quantity. For instance, what sort of thing a dynamical law of nature must be to support such dependence? This is an extremely interesting question, and one I think is worth pursuing, but not here.

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