Class 10 Proof Strategies in TFL

Zee R. Perry

Let's do some logic

Diagnosing an Incorrect "Proof"

First Incorrect "Proof":

1	$(\neg L \wedge A) \vee L$	Pr.
2	$\quad \neg L \wedge A$	Ass.
3	$\quad \neg L$	$\wedge$E, 3
4	$\quad A$	$\wedge$E, 1
5	$\quad L$	Ass.
6	$\quad \bot$	$\neg$E, 3,5
7	$\quad A$	Ex, 6
8	$\quad A$	$\vee$E, 1, 2–4, 5–7

Diagnosing an Incorrect "Proof"

Second Incorrect "Proof":

1	$A \wedge (B \wedge C)$	Pr.
2	$(B \vee C) \rightarrow D$	Pr.
3	B	$\wedge$E, 1
4	$B \vee C$	$\vee$I, 3
5	D	$\rightarrow$E, 4, 2

Proof without Rules & Line Numbers

1	$P \wedge S$	____ , ____
2	$S \rightarrow R$	____ , ____
3	P	____ , ____
4	S	____ , ____
5	R	____ , ____
6	$R \vee E$	____ , ____

1	$A \rightarrow D$	____ , ____
2	$A \wedge B$	____ , ____
3	A	____ , ____
4	D	____ , ____
5	$D \vee E$	____ , ____
6	$(A \wedge B) \rightarrow (D \vee E)$	____ , ____

Proof without Rules & Line Numbers

1	$\neg L \to (J \vee L)$	____ , ____
2	$\neg L$	____ , ____
3	$J \vee L$	____ , ____
4	J	____ , ____
5	$J \wedge J$	____ , ____
6	J	____ , ____
7	L	____ , ____
8	$\bot$	____ , ____
9	J	____ , ____
10	J	____ , ____

Rule: Reiteration (“Reit.” or “R”)

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. P	1. $(A \leftrightarrow G)$	1. I′m busy
$\therefore$ P	$\therefore$ $(A \leftrightarrow G)$	$\therefore$ I′m busy

Reiteration Rule, “Reit.”

	⋮	
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{A}$	Reit., n

Rule: Conjunction Introduction (“$\wedge$I” or “$\wedge$In”)

1. P
2. $Q \vee P$

$\therefore\ P \wedge (Q \vee P)$

1. $(A \leftrightarrow G)$
2. $\neg S$

$\therefore\ (A \leftrightarrow G) \wedge \neg S$

1. There's a giraffe in this building
2. Rutgers is in New Jersey

∴ There's a giraffe in this building *and* Rutgers is in NJ

Conjunction Introduction, “$\wedge$I”

n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
u	$\mathscr{A} \wedge \mathscr{B}$	$\wedge$I , n, m

Rule: Conjunction Elimination ("$\wedge$E" or "$\wedge$Out")

1. $P \wedge (Q \vee P)$

$\therefore\ Q \vee P$

1. $\neg S \wedge (A \leftrightarrow G)$

$\therefore\ \neg S$

1. There's a giraffe in this building *and* Rutgers is in NJ

$\therefore$ There's a giraffe in this building

Conjunction Elimination, "$\wedge$E"				
n	$\mathscr{A} \wedge \mathscr{B}$	(Some Rule)	,	(Line Numbers)
$\vdots$	$\vdots$	$\vdots$		
u	$\mathscr{A}$	$\wedge$E	,	n
$\vdots$	$\vdots$	$\vdots$		
v	$\mathscr{B}$	$\wedge$E	,	n

Rule: Disjunction Introduction (“$\vee$I” or “$\vee$In”)

1. It's Wednesday.

$\therefore$ Either it's Wednesday or I'm a kettle of fish.

1. $\neg S \wedge A$

$\therefore$ $(\neg S \wedge A) \vee B$

1. P

$\therefore$ $P \vee Q$

Disjunction Introduction, “$\vee$I”

For **ANY** “$\mathscr{B}$” !

n	$\mathscr{A}$	(Some Rule) , (Line Numbers)	
$\vdots$	$\vdots$	$\vdots$	
u	$\mathscr{A} \vee \mathscr{B}$	$\vee$I ,	n

Rule: Conditional Elimination ("→E" or "→Out")

1. If it's raining then my hair will get wet.
2. It's raining.

∴ My hair will get wet

1. $(\neg S \wedge A) \rightarrow (B \leftrightarrow A)$
2. $\neg S \wedge A$

$\therefore\ B \leftrightarrow A$

1. $P \rightarrow Q$
2. P

$\therefore\ Q$

Conditional Elimination, "→E"

n	$\mathscr{A} \rightarrow \mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
u	$\mathscr{B}$	→E , n,m

Rules that involve SUB-PROOFS

There are two kinds of rules in Natural Deduction for TFL:

Rules that care about previous lines in the *main* proof

- Reit
- $\wedge$E
- $\wedge$I
- $\vee$I
- $\rightarrow$E
- $\leftrightarrow$E

Rules that *also* care about previous *sub-proofs*

- $\rightarrow$I
- $\leftrightarrow$I
- $\vee$E
- $\neg$I
- $\neg$E
- IP
- Ex

What is a sub-proof? Roughly: You are allowed to start a *new* proof **within** your current proof, and these rules will depend on there being

Rule: Conditional Introduction ("$\to$I" or "$\to$In")

Conditional Introduction is a rule that lets you write down a **conditional** statement (i.e. a sentence whose main connective is '$\to$'), given some lines earlier in your proof. What would this even look like? What—*IN GENERAL*—is required for "$P \to Q$" to be true?

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.

But it also *doesn't require P to actually be true*! So how could we make sense of this in our proof system?

Rule: Conditional Introduction ("$\to$I" or "$\to$In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.
But it also *doesn't require* P *to actually be true*! So how could we make sense of this in our proof system?

Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.
But it also *doesn't require* P *to actually be true*! So how could we make sense of this in our proof system?

Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

Second Pass: So "$P \to Q$" is true whenever $\dfrac{\textbf{1.}\ P}{\therefore\ Q}$ is valid.

Rule: Conditional Introduction ("$\to$I" or "$\to$In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.
But it also *doesn't require P to actually be true*! So how could we make sense of this in our proof system?

Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

Second Pass: So "$P \to Q$" is true whenever $\dfrac{\textbf{1.}\ P}{\therefore\ Q}$ is valid.
But how could we—while in the middle of a proof!—show that some *other* argument is valid??

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.

But it also *doesn't require P to actually be true*! So how could we make sense of this in our proof system?

Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

Second Pass: So "$P \to Q$" is true whenever $\dfrac{\textbf{1.}\ \ P}{\therefore\ \ Q}$ is valid.

But how could we—while in the middle of a proof!—show that some *other* argument is valid??

Simplest Solution: Put a proof on your proof so you can prove while you prove!

Rule: Conditional Introduction ("→I" or "→In")

What—*IN GENERAL*—is required for "$P \to Q$" to be true?

First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.
But it also *doesn't require P to actually be true*! So how could we make sense of this in our proof system?

Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

Second Pass: So "$P \to Q$" is true whenever $\dfrac{\textbf{1.}\ \ P}{\therefore\ \ Q}$ is valid.
But how could we—while in the middle of a proof!—show that some *other* argument is valid??

Simplest Solution: Put a proof on your proof so you can prove while you prove! So how the heck would *that* work?!

Sub-proofs and Assumptions

Within a proof, you can start **another** proof once you've moved past the premises stage:

1	$A \rightarrow C$	Pr.

Sub-proofs and Assumptions

Within a proof, you can start **another** proof once you've moved past the premises stage:

1	$A \to C$	Pr.
2	$C \to B$	Pr.

Sub-proofs and Assumptions

Within a proof, you can start **another** proof once you've moved past the premises stage:

1	$A \to C$	Pr.
2	$C \to B$	Pr.

WTS: $A \to B$

Sub-proofs and Assumptions

When you start a sub-proof, you introduce ***Assumptions*** instead of premises.

1	$A \rightarrow C$	Pr.
2	$C \rightarrow B$	Pr.
WTS:	$A \rightarrow B$	
3		

Sub-proofs and Assumptions

1	$A \rightarrow C$	Pr.
2	$C \rightarrow B$	Pr.
WTS:	$A \rightarrow B$	
3	A	Ass.

When you start a sub-proof, you introduce ***Assumptions*** instead of premises.

Sub-proofs and Assumptions

1	$A \rightarrow C$	Pr.
2	$C \rightarrow B$	Pr.
WTS:	$A \rightarrow B$	
3	A	Ass.
4		

When you start a sub-proof, you introduce ***Assumptions*** instead of premises.

Any line you add *within* your sub-proof can reference any line above it, ***including lines from your main proof!***

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	A	Ass.
4	C	$\to$E, 1, 3

Any line you add *within* your sub-proof can reference any line above it***, including lines from your main proof!***

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	A	Ass.
4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4

Any line you add *within* your sub-proof can reference any line above it, ***including lines from your main proof!***

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	A	Ass.
4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4
6		

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	$\quad A$	Ass.
4	$\quad C$	$\to$E, 1, 3
5	$\quad B$	$\to$E, 2, 4
6	B	Reit, 5

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

Sub-proofs and Assumptions

1	$A \rightarrow C$	Pr.	
2	$C \rightarrow B$	Pr.	
WTS:	$A \rightarrow B$		
3	A	Ass.	
4	C	$\rightarrow$E, 1, 3	
5	B	$\rightarrow$E, 2, 4	
6			

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

Sub-proofs and Assumptions

1		$A \to C$	Pr.
2		$C \to B$	Pr.
WTS:	$A \to B$		
3		A	Ass.
4		C	$\to$E, 1, 3
5		B	$\to$E, 2, 4
6			

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

However, there are some rules that can only be applied *if a sub-proof has occurred earlier in the proof*! One such rule is $\to$I

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	A	Ass.
4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4
6		$\to$I, 3–5

However, there are some rules that can only be applied *if a sub-proof has occurred earlier in the proof*! One such rule is $\to$I

Sub-proofs and Assumptions

1	$A \rightarrow C$	Pr.
2	$C \rightarrow B$	Pr.
WTS:	$A \rightarrow B$	
3	A	Ass.
4	C	$\rightarrow$E, 1, 3
5	B	$\rightarrow$E, 2, 4
6	$A \rightarrow B$	$\rightarrow$I, 3–5

However, there are some rules that can only be applied *if a sub-proof has occurred earlier in the proof*! One such rule is $\rightarrow$I

Rule: Conditional Introduction ("→I" or "→In")

Conditional Introduction appeals to the following kind of reasoning:

Rule: Conditional Introduction ("→I" or "→In")

Conditional Introduction appeals to the following kind of reasoning:

Since $\dfrac{\textbf{1.}\ P}{\therefore\ Q}$ is a valid argument, it must be true that $P \rightarrow Q$

Rule: Conditional Introduction ("$\rightarrow$I" or "$\rightarrow$In")

Conditional Introduction appeals to the following kind of reasoning:

Since $\dfrac{\textbf{1.}\ \ P}{\therefore\ \ Q}$ is a valid argument, it must be true that $P \rightarrow Q$

If we assume that it'll rain tonight, then it follows that I'll get my hair wet. So, *no matter what* the weather's like, we know that the conditional "If it rains, then I'll get my hair wet" is true.

Rule: Conditional Introduction ("→I" or "→In")

Conditional Introduction, "→I"

⋮	⋮	⋮
n	| $\mathscr{A}$	Ass.
⋮	| ⋮	⋮
u	| $\mathscr{B}$	(Some Rule) , (Line Numbers)
1		
⋮	⋮	⋮
v	$\mathscr{A} \rightarrow \mathscr{B}$	→I , n–m

Rule: Disjunction Elimination (“$\vee$E” or “$\vee$Out”)

Disjunction Elimination *also* relies on the use of sub-proofs!

Rule: Disjunction Elimination ("$\vee$E" or "$\vee$Out")

Disjunction Elimination *also* relies on the use of sub-proofs!

1. *X* is either in the library or in the student center.

2. If *X* is in the library, then *X* is on College Ave.

3. If *X* is in the student center, then *X* is on College Ave.

$\therefore$ *X* is on College Ave!

Rule: Disjunction Elimination (“$\lor$E” or “$\lor$Out”)

Disjunction Elimination, “$\lor$E”

i	$\mathscr{A} \lor \mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
n	⎸ $\mathscr{A}$	Ass.
m	⎸ $\mathscr{C}$	(Some Rule) , (Line Numbers)
⋮		⋮
u	⎸ $\mathscr{B}$	Ass.
v	⎸ $\mathscr{C}$	(Some Rule) , (Line Numbers)
j	$\mathscr{C}$	$\lor$E , i, n–m, u–v

Rule: Negation Elimination ("¬E" or "𝕏In" or "⊥In")

Negation Elimination is also sometimes known as Contradiction Introduction, because it only "eliminates" negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

Rule: Negation Elimination (“¬E” or “⨳In” or “⊥In”)

Negation Elimination is also sometimes known as Contradiction Introduction, because it only “eliminates” negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

Here’s what I mean:

Rule: Negation Elimination (“¬E” or “𝕏In” or “⊥In”)

Negation Elimination is also sometimes known as Contradiction Introduction, because it only “eliminates” negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

Negation Elimination, “¬E”		
⋮	⋮	⋮
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\neg\mathscr{A}$	(Some Rule) , (Line Numbers)
j	$\bot$	¬E , n, m

Rule: Negation Elimination ("¬E" or "$\mathbb{X}$In" or "⊥In")

Negation Elimination is also sometimes known as Contradiction Introduction, because it only "eliminates" negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

Negation Elimination, "¬E"		
⋮	⋮	⋮
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\neg\mathscr{A}$	(Some Rule) , (Line Numbers)
j	$\mathbb{X}$	¬E , n, m

Rule: Negation Elimination (“¬E” or “𝕏In” or “⊥In”)

Negation Elimination is also sometimes known as Contradiction Introduction, because it only “eliminates” negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

Negation Elimination, “¬E”		
$\vdots$	$\vdots$	$\vdots$
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
$\vdots$	$\vdots$	$\vdots$
m	$\neg\mathscr{A}$	(Some Rule) , (Line Numbers)
j	$\bot$	¬E , n, m

Rule: Negation Introduction ("¬I" or "¬In")

Negation introduction is a way of introducing a negation by "eliminating" a contradiction (when that contradiction arises **IN A SUB-PROOF!**). Specifically, if a sub-proof leads you to a contradiction, then you ***know*** that the assumptions you made *cannot* be true!

Rule: Negation Introduction ("¬I" or "¬In")

Negation introduction is a way of introducing a negation by "eliminating" a contradiction (when that contradiction arises **IN A SUB-PROOF!**). Specifically, if a sub-proof leads you to a contradiction, then you ***know*** that the assumptions you made *cannot* be true!

Here's what I mean:

Rule: Negation Introduction (“¬I” or “¬In”)

Negation introduction is a way of introducing a negation by “eliminating” a contradiction (when that contradiction arises **IN A SUB-PROOF!**). Specifically, if a sub-proof leads you to a contradiction, then you ***know*** that the assumptions you made *cannot* be true!

Negation Introduction, “¬I”		
⋮	⋮	⋮
n	$\quad\mathscr{A}$	Ass.
⋮	$\quad$⋮	⋮
m	$\quad\bot$	(Some Rule) , (Line Numbers)
j	$\neg\mathscr{A}$	¬I , n–m

Rule: Indirect Proof ("IP")

Indirect Proof, also called "Proof by Contradiction", is extremely similar to negation introduction. However, instead of *introducing* a negation at the end, we're *assuming* a negated statement, in order to show that this leads to a contradiction. If assuming that "*P*" is false leads to a contradiction, then it must* be that *P* is true!

Rule: Indirect Proof ("IP")

Indirect Proof, also called "Proof by Contradiction", is extremely similar to negation introduction. However, instead of *introducing* a negation at the end, we're *assuming* a negated statement, in order to show that this leads to a contradiction. If assuming that "*P*" is false leads to a contradiction, then it must* be that *P* is true!

Indirect Proof, "IP"

⋮	⋮	⋮
n	$\neg\mathscr{A}$	Ass.
⋮	⋮	⋮
m	$\bot$	(Some Rule) , (Line Numbers)
j	$\mathscr{A}$	IP , n–m

Rule: Explosion (“Ex” or “⊥E” or “XOut”)

The Explosion rule has a fun name because it describes a really wild feature of TFL: Literally anything follows from a contradiction. That is, any argument with a contradiction among its premises is valid, no matter what else is in the argument!

Rule: Explosion ("Ex" or "⊥E" or "XOut")

The Explosion rule has a fun name because it describes a really wild feature of TFL: Literally anything follows from a contradiction. That is, any argument with a contradiction among its premises is valid, no matter what else is in the argument!

Explosion, "Ex"

n	$\bot$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{A}$	Ex , n

The rest of the Homework!

Chapter 15 Block C

Proofs 1, 2, 3, and 4.

1. $J \rightarrow \neg J \therefore \neg J$
2. $Q \rightarrow (Q \wedge \neg Q) \therefore \neg Q$
3. $A \rightarrow (B \rightarrow C) \therefore (A \wedge B) \rightarrow C$
4. $K \wedge L \therefore K \leftrightarrow L$

How to Construct Good Proofs!

Work backwards from what you **want**

* More often than not, there are only a few rules that will be able to output a sentence like your desired conclusion.
* If you arrived at your concluding line using one of those rules, what other lines would the **rest** of your proof have to contain?

Work forwards from what you **have**

* If you've got some starting sentences, there are usually a few lines that you can do just by figuring out what you can extract from them.
* That is, use any elimination rules that are appropriate to the main connective of your premises, and "pull out" whatever you can into its own line.

How to Construct Good Proofs!

Give yourself mini-goals (but don't lose sleep if you miss 'em)

* When working backwards from your desired conclusion, make your mini-goal those sentences required to get to that point.
* When working forwards from your existing premises/lines, you might find a rule that you cannot use to work forward, because it requires a line you don't have. Make your mini-goal the lines that will allow you to use more of those rules.

Having trouble? Try Proof by Contradiction ("IP")

* If you're not sure of how to proceed, whether because you ran out of ways to work forward/back or because you're stumped, try this:
* Start a sub-proof, and assume **the negation of** your conclusion! Then, saying within that sub-proof, try to prove a contradiction ("$\bot$").

Proof Practice!

Let's do some proofs together!

$\neg(P \wedge \neg P)$

$\neg\neg A \to A$

$\neg A \to (A \to \bot)$

$(A \vee \neg B) \to (B \to A)$