Class 11 Proof Strategies and Proof-Theoretic Concepts

Zee R. Perry

Let's do some logic

Core Rules of Natural Deduction for TFL

Rules that rely on previous lines in the *main* proof

Reit	Reiteration
$\wedge$I	Conjunction Introduction
$\wedge$E	Conjunction Elimination
$\vee$I	Disjunction Introduction
$\rightarrow$E	Conditional Elimination
$\leftrightarrow$E	Biconditional Elimination

Rules that *also* (generally) occur *within* or care *about* sub-proofs

$\rightarrow$I	Conditional Introduction
$\leftrightarrow$I	Biconditional Introduction
$\vee$E	Disjunction Elimination
$\neg$E	Negation Elimination
$\neg$I	Negation Introduction
IP	Indirect Proof
Ex	Explosion

The rest of the Homework!

Chapter 15 Block C

Proofs 1, 2, 3, and 4.

1. $J \to \neg J \therefore \neg J$
2. $Q \to (Q \land \neg Q) \therefore \neg Q$
3. $A \to (B \to C) \therefore (A \land B) \to C$
4. $K \land L \therefore K \leftrightarrow L$

The rest of the Homework!

Chapter 15 Block C

Proofs 1, 2, 3, and 4.

1. $J \to \neg J \therefore \neg J$
2. $Q \to (Q \land \neg Q) \therefore \neg Q$
3. $A \to (B \to C) \therefore (A \land B) \to C$
4. $K \land L \therefore K \leftrightarrow L$

But how do we even *start* to approach things like these?!

When Constructing Proofs, Give Yourself Mini-Goals

What should these goals be? There are two key methods:

When Constructing Proofs, Give Yourself Mini-Goals

What should these goals be? There are two key methods:

Work backwards from what you want

* More often than not, there are only a few rules that will be able to output a sentence like your desired conclusion.

When Constructing Proofs, Give Yourself Mini-Goals

What should these goals be? There are two key methods:

Work backwards from what you **want**

* More often than not, there are only a few rules that will be able to output a sentence like your desired conclusion.
* If you arrived at your concluding line using one of those rules, what other lines would the **rest** of your proof have to contain?

When Constructing Proofs, Give Yourself Mini-Goals

What should these goals be? There are two key methods:

Work backwards from what you **want**

* More often than not, there are only a few rules that will be able to output a sentence like your desired conclusion.
* If you arrived at your concluding line using one of those rules, what other lines would the **rest** of your proof have to contain?

Work forwards from what you **have**

* If you've got some starting premises, there are usually a few lines you can do just by figuring out what you can extract from them.

When Constructing Proofs, Give Yourself Mini-Goals

What should these goals be? There are two key methods:

Work backwards from what you **want**

* More often than not, there are only a few rules that will be able to output a sentence like your desired conclusion.
* If you arrived at your concluding line using one of those rules, what other lines would the **rest** of your proof have to contain?

Work forwards from what you **have**

* If you've got some starting premises, there are usually a few lines you can do just by figuring out what you can extract from them.
* That is, use any elimination rules that are appropriate to the main connective of your premises, and "pull out" whatever you can into its own line.

Core Rules of Natural Deduction for TFL

Rules that rely on previous lines in the *main* proof

Reit	Reiteration
∧I	Conjunction Introduction
∧E	Conjunction Elimination
∨I	Disjunction Introduction
→E	Conditional Elimination
↔E	Biconditional Elimination

Rules that *also* (generally) occur *within* or care *about* sub-proofs

→I	Conditional Introduction
↔I	Biconditional Introduction
∨E	Disjunction Elimination
¬E	Negation Elimination
¬I	Negation Introduction
IP	Indirect Proof
Ex	Explosion

Core Rules of Natural Deduction for TFL

Look at the main connectives of the sentences that you **have/want!**

Rules that rely on previous lines in the *main* proof

Reit	Reiteration
$\wedge$**I**	Conjunction Introduction
$\wedge$**E**	Conjunction Elimination
$\vee$**I**	Disjunction Introduction
$\rightarrow$**E**	Conditional Elimination
$\leftrightarrow$**E**	Biconditional Elimination

Rules that *also* (generally) occur *within* or care *about* sub-proofs

$\rightarrow$**I**	Conditional Introduction
$\leftrightarrow$**I**	Biconditional Introduction
$\vee$**E**	Disjunction Elimination
$\neg$**E**	Negation Elimination
$\neg$**I**	Negation Introduction
IP	Indirect Proof
Ex	Explosion

Core Rules of Natural Deduction for TFL

Look at the main connectives of the sentences that you **have/want**! What's required for you to apply the rule that **Eliminates/Introduces** that connective? Can you do it right away?

Rules that rely on previous lines in the *main* proof		**Rules that *also* (generally) occur *within* or care *about* sub-proofs**	
Reit	Reiteration	$\rightarrow$**I**	Conditional Introduction
$\wedge$**I**	Conjunction Introduction	$\leftrightarrow$**I**	Biconditional Introduction
$\wedge$**E**	Conjunction Elimination	$\vee$**E**	Disjunction Elimination
$\vee$**I**	Disjunction Introduction	$\neg$**E**	Negation Elimination
$\rightarrow$**E**	Conditional Elimination	$\neg$**I**	Negation Introduction
$\leftrightarrow$**E**	Biconditional Elimination	**IP**	Indirect Proof
		Ex	Explosion

How to Construct Good Proofs (in Summary)

Give yourself mini-goals (but don′t lose sleep if you miss ′em)

* When working **backwards** from your desired conclusion, make your mini-goal those sentences required to get to that point.
* When working **forwards** from your existing premises/lines, you might find a rule that you cannot use to work forward, because it requires a line you don′t have. Make your mini-goal to write down the lines that will allow you to use one of those rules.

How to Construct Good Proofs (in Summary)

Give yourself mini-goals (but don't lose sleep if you miss 'em)

* When working **backwards** from your desired conclusion, make your mini-goal those sentences required to get to that point.
* When working **forwards** from your existing premises/lines, you might find a rule that you cannot use to work forward, because it requires a line you don't have. Make your mini-goal to write down the lines that will allow you to use one of those rules.

Still having trouble? Try Proof by Contradiction/("IP")

* If you're not sure of how to proceed, whether because you ran out of ways to work forward/back or because you're stumped, try this:
* Make a sub-proof & assume **the negation of** your conclusion! Then, staying within that sub-proof, try to prove a contradiction ("$\bot$").

Proof Practice!

Let's do some proofs together, using these strategies!

First one: $A \vee B, A \rightarrow C, B \rightarrow D \therefore C \vee D$

Proof Practice!

Let's do some proofs together, using these strategies!

First one: $A \vee B, A \rightarrow C, B \rightarrow D \therefore C \vee D$

Optional ones: $A \rightarrow B, A \rightarrow C \therefore A \rightarrow (B \wedge C)$

$(A \wedge B) \rightarrow C \therefore A \rightarrow (B \wedge C)$

Remember this semantic concept?

Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: “$\models$”

Remember this semantic concept?

Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: "$\vDash$"

Reminder: Meaning of the Double Turnstile "$\vDash$"

Instead of saying:

- "The sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' together ENTAIL the sentence '$\mathscr{P}$'."
- "There's NO POSSIBLE CASE where all the sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', are true and the sentence '$\mathscr{P}$' is false."

Remember this semantic concept?

Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: "$\vDash$"

Reminder: Meaning of the Double Turnstile "$\vDash$"

Instead of saying:

- "The sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' together ENTAIL the sentence '$\mathscr{P}$'."
- "There's NO POSSIBLE CASE where all the sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', are true and the sentence '$\mathscr{P}$' is false."

We can say: "$\mathscr{A}, \mathscr{B}, \mathscr{C} \vDash \mathscr{P}$"

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, "$\vdash$"! It is specifically focused on **what can be proved** within our system:

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, "$\vdash$"! It is specifically focused on **what can be proved** within our system:

Meaning of the *Single* Turnstile "$\vdash$"

Instead of saying:

- "From premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' you can PROVE the sentence '$\mathscr{P}$'."

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, "$\vdash$"! It is specifically focused on **what can be proved** within our system:

Meaning of the *Single* Turnstile "$\vdash$"

Instead of saying:

- "From premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' you can PROVE the sentence '$\mathscr{P}$'."
- "There exists at least one PROOF that (1) follows all the formal rules, (2) has premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', and (3) has, **as its final line**, the sentence '$\mathscr{P}$'."

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, "$\vdash$"! It is specifically focused on **what can be proved** within our system:

Meaning of the *Single* Turnstile "$\vdash$"

Instead of saying:

- "From premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' you can PROVE the sentence '$\mathscr{P}$'."
- "There exists at least one PROOF that (1) follows all the formal rules, (2) has premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', and (3) has, **as its final line**, the sentence '$\mathscr{P}$'."

We can say: "$\mathscr{A}, \mathscr{B}, \mathscr{C} \vdash \mathscr{P}$"

Some Proof-theoretic concepts!

The Double Turnstile, "$\vDash$", is about *the existence of* VALUATIONS.

I.e. it's about what sentences can be true and false at the same time (in, e.g., a given row of a truth-table).

The Single Turnstile, "$\vdash$", is about *the existence of* PROOFS.

I.e. it's about whether there's a way to construct a formal proof that has those sentences located at specific lines.

Things we can define using '⊢'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

Things we can define using '$\vdash$'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

$$\vdash \mathscr{A}$$

Things we can define using '$\vdash$'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

$$\vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved using *no* premises at all!

Things we can define using '$\vdash$'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

$$\vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved using *no* premises at all!

When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

Things we can define using '$\vdash$'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

$$\vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved using *no* premises at all!

When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{and} \qquad \mathscr{B} \vdash \mathscr{A}$$

Things we can define using '$\vdash$'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

$$\vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved using *no* premises at all!

When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{\textit{and}} \qquad \mathscr{B} \vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved from $\mathscr{B}$, ***and* vice versa**!

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

Things we can define using '⊢'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

This means you can **prove a contradiction** from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

This means you can **prove a contradiction** from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

When two sentences are "Provably Consistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Consistent" is just to say that they're ***NOT*** provably *in*consistent, that is:

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

This means you can **prove a contradiction** from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

When two sentences are "Provably Consistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Consistent" is just to say that they're ***NOT*** provably *in*consistent, that is:

" **It's not the case that** $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$ "

Things we can define using '$\vdash$'

When sentences are "PROVABLY INCONSISTENT"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are PROVABLY INCONSISTENT" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

This means you can **prove a contradiction** from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

When two sentences are "PROVABLY CONSISTENT"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are PROVABLY CONSISTENT" is just to say that they're ***NOT*** provably *in*consistent, that is:

" **It's not the case that** $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$ "

or

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \cdots \nvdash \bot$$

Things we can define using '$\vdash$'

When sentences are "PROVABLY INCONSISTENT"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are PROVABLY INCONSISTENT" is just to say:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$$

This means you can **prove a contradiction** from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

When two sentences are "PROVABLY CONSISTENT"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are PROVABLY CONSISTENT" is just to say that they're ***NOT*** provably *in*consistent, that is:

" **It's not the case that** $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$ "

or

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \cdots \nvdash \bot$$

This means you ***can't*** *prove a contradiction* from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

Sometimes you *have* to work backwards!

$\neg(P \land \neg P)$

$\neg\neg A \to A$

$\neg A \to (A \to \bot)$

$(A \lor \neg B) \to (B \to A)$

Sometimes you *have* to work backwards!

$\neg(P \wedge \neg P)$

$\neg\neg A \to A$

$\neg A \to (A \to \bot)$

$(A \vee \neg B) \to (B \to A)$

Technically, you can still understand this as "proving that an argument is valid," it's just that the argument we're concerned with has no premises:

i.e. It's one we'd write as " $\therefore \mathscr{A}$" (for some sentence, $\mathscr{A}$).

Sometimes you *have* to work backwards!

$\neg(P \wedge \neg P)$ $\neg A \rightarrow (A \rightarrow \bot)$

$\neg\neg A \rightarrow A$ $(A \vee \neg B) \rightarrow (B \rightarrow A)$

Technically, you can still understand this as "proving that an argument is valid," it's just that the argument we're concerned with has no premises:

i.e. It's one we'd write as " $\therefore \mathscr{A}$" (for some sentence, $\mathscr{A}$).

To prove a sentence from no premises is just to prove that it is a **theorem** of TFL. I.e. you're showing that: $\vdash \mathscr{A}$

Sometimes you *have* to work backwards!

$\neg(P \wedge \neg P)$

$\neg\neg A \rightarrow A$

$\neg A \rightarrow (A \rightarrow \bot)$

$(A \vee \neg B) \rightarrow (B \rightarrow A)$

Homeworks!

Chapter 16

Block A: Question 1

Block D: Question 1

Chapter 17

Block A: Fill-in Proofs 1 and 2

YOU *WILL* NEED TO USE THE EXTRA RULES IN CHAPTER 17!!

Block B: Question 1