Class 12 - Soundness and Completeness of TFL

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Let's do some logic!

First, some Admin: Midterm Next Class!

This coming Wednesday is the **Second Midterm Exam!!** The exam will chapters 15 to 20, with a focus on 15, 16, and 18 (where the majority of our homeworks came from).

You will be permitted to ***create your own*** single-side 8.5×11″ (or, preferably, a *double*-sided *half*-sheet of 8.5×11″ paper) "cheat" sheet.

ALSO! you will be permitted to bring and make use of (*either, or both, of*) my little "Basic Deduction Rules for TFL" half-sheets.

First, some Admin: Midterm Next Class!

Content-wise, you will need to know:

Semantic Concepts: What a *Tautology* is.
What a *Contradiction* is.
What it means for two sentences to be *logically equivalent*.

Proof Theory: How to correctly *format* proofs.
How to *apply deductive rules* to write down new lines in proofs.
How to interpret *schematic definitions* of rules (as written on my little rules-guide half-sheets).
How to *apply* our three *proof-strategies* (Working Forward, Working Backwards, and Indirect Proof) to write your own proofs or complete an already-started proof.

Proof-Theoretic Concepts: What a *Theorem (of TFL)* is.
What it means for two sentences to be *provably equivalent*.

$$A \to B, A \to C \vdash A \to (B \land C)$$

$$A \to B, A \to C \vdash A \to (B \land C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \land C)$	
3		
		??
	$A \to (B \land C)$	

$$A \to B, A \to C \vdash A \to (B \land C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \land C)$	
3	A	Ass.
	$B \land C$	??
		??
	$A \to (B \land C)$	$\to$I, ??–??

$$A \to B, A \to C \vdash A \to (B \wedge C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \wedge C)$	
3	A	Ass.
	$B \wedge C$	??∧I,
		??
	$A \to (B \wedge C)$	→I, ??–??

$$A \to B, A \to C \vdash A \to (B \wedge C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \wedge C)$	
3	A	Ass.
4	B	$\to$E, 1, 3
	$B \wedge C$	??$\wedge$I,
		??
	$A \to (B \wedge C)$	$\to$I, ??–??

$$A \to B, A \to C \vdash A \to (B \land C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \land C)$	
3	A	Ass.
4	B	$\to$E, 1, 3
5	C	$\to$E, 2, 3
	$B \land C$	??$\land$I,
		??
	$A \to (B \land C)$	$\to$I, ??–??

$$A \to B, A \to C \vdash A \to (B \wedge C)$$

1	$A \to B$	Pr.
2	$A \to C$	Pr.
WTS:	$A \to (B \wedge C)$	
3	A	Ass.
4	B	→E, 1, 3
5	C	→E, 2, 3
6	$B \wedge C$	∧I,4, 5
		??
	$A \to (B \wedge C)$	→I, ??–??

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2	$A \to C$	Pr.
WTS:	$A \to (B \wedge C)$	
3	A	Ass.
4	B	$\to$E, 1, 3
5	C	$\to$E, 2, 3
6	$B \wedge C$	$\wedge$I,4, 5
		??
	$A \to (B \wedge C)$	$\to$I, 3–6

$$\neg\neg A \rightarrow A$$

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WTS: $\neg\neg A \rightarrow A$

1	$\quad \neg\neg A$	Ass.
2	$\quad\quad \neg A$	Ass.
3	$\quad\quad \bot$	$\neg$E, 1, 2
4	$\quad A$	IP, 2–3
5	$\neg\neg A \rightarrow A$	$\rightarrow$I, 1–4

Filling-in Citations Proofs

1	$W \to \neg B$	Pr.
2	$A \wedge W$	Pr.
3	$B \vee (J \wedge K)$	Pr.
4	W	
5	$\neg B$	
6	$\neg B$	
7	$J \wedge K$	
8	K	

Filling-in Citations Proofs

1	$W \rightarrow \neg B$	Pr.
2	$A \wedge W$	Pr.
3	$B \vee (J \wedge K)$	Pr.
4	W	
5	$\neg B$	
6	$\neg B$	
7	$J \wedge K$	
8	K	

1	$L \leftrightarrow \neg O$	Pr.
2	$L \vee \neg O$	Pr.
3	$\quad \neg L$	
4	$\quad \neg O$	
5	$\quad L$	
6	$\quad \bot$	
7	$\neg\neg L$	
8	L	

$$E \vee F, F \vee G, \neg F \therefore E \wedge G$$

Any questions about this one? If not I'll move on.

Semantics and Proof-theory Collide!

We've covered ***multiple*** definitions of very similar concepts:

Remember this semantic concept?

Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: “$\vDash$”

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Reminder: Meaning of the Double Turnstile "$\vDash$"

Instead of saying:

- "The sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' together ENTAIL the sentence '$\mathscr{P}$'."
- "There's NO POSSIBLE CASE where all the sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', are true and the sentence '$\mathscr{P}$' is false."

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We can say: "$\mathscr{A},\mathscr{B},\mathscr{C} \vDash \mathscr{P}$"

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, "$\vdash$"! It is specifically focused on **what can be proved** within our system:

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Meaning of the *Single* Turnstile "$\vdash$"

Instead of saying:

- "From premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' you can PROVE the sentence '$\mathscr{P}$'."
- "There exists at least one PROOF that (1) follows all the formal rules, (2) has premises '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', and (3) has, **as its final line**, the sentence '$\mathscr{P}$'."

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We can say: " $\mathscr{A}, \mathscr{B}, \mathscr{C} \vdash \mathscr{P}$ "

Some Proof-theoretic concepts!

The Double Turnstile, "$\vDash$", is about *the existence of* VALUATIONS.

I.e. it's about what sentences can be true and false at the same time (in, e.g., a given row of a truth-table).

The Single Turnstile, "$\vdash$", is about *the existence of* PROOFS.

I.e. it's about whether there's a way to construct a formal proof that has those sentences located at specific lines.

Things we can define using '⊢'

When a sentence is a "Theorem of TFL"

To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

Things we can define using '$\vdash$'

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This means that $\mathscr{A}$ can be proved using *no* premises at all!

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To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

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When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{and} \qquad \mathscr{B} \vdash \mathscr{A}$$

Things we can define using '$\vdash$'

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To say that "Sentence '$\mathscr{A}$' is a Theorem of TFL" is just to say:

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To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{\textit{and}} \qquad \mathscr{B} \vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved from $\mathscr{B}$, ***and*** **vice versa**!

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

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To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are PROVABLY CONSISTENT" is just to say that they're ***NOT*** provably *in*consistent, that is:

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" **It's not the case that** $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$ "

Things we can define using '$\vdash$'

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" **It's not the case that** $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots \vdash \bot$ "

or

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \cdots \nvdash \bot$$

Things we can define using '$\vdash$'

When sentences are "Provably Inconsistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Inconsistent" is just to say:

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When two sentences are "Provably Consistent"

To say that "Sentences '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc. are Provably Consistent" is just to say that they're ***NOT*** provably *in*consistent, that is:

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or

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \cdots \nvdash \bot$$

This means you ***can't*** *prove a contradiction* from '$\mathscr{A}$', '$\mathscr{B}$', '$\mathscr{C}$', etc.

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row in that sentence's truth-table is a "F"

Logically Equivalent: These sentences always have the *same* truth-value for each row of their shared truth-table.

Logically Inconsistent: There's no row of their shared truth table where both sentences are true.

Valid$_{\textit{Semantic}}$**:** In *ALL* the rows where every premise is assigned a "T", the conclusion is also assigned a "T".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction$_{\textit{Proof-Theoretic}}$**:** You can prove the sentence's *negation* from no premises.

Provably Equivalent: You can prove one sentence using the other as your premise, *and* vice versa.

Provably Inconsistent: You can prove a contradiction using those sentences as two premises.

Valid$_{\textit{Proof-Theoretic}}$**:** There exists a proof from those premises to that conclusion.

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row of that sentence's *negation*'s truth table is "T".

Logically Equivalent: The biconditional between those two sentences is a *tautology*

Logically Inconsistent: The conjunction of those two sentences is a *contradiction*.

Valid$_{Semantic}$**:** There is ***NO*** row where all the premises are assigned "T" and the conclusion is assigned an "F".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction$_{Proof\text{-}Theoretic}$**:** You can prove the sentence's *negation* from no premises.

Provably Equivalent: You can prove one sentence using the other as your premise, *and* vice versa.

Provably Inconsistent: You can prove a contradiction using those sentences as two premises.

Valid$_{Proof\text{-}Theoretic}$**:** There exists a proof from those premises to that conclusion.

Semantic Concepts:

Tautology: $\vDash A$

Contradiction: $\vDash \neg A$

Logically Equivalent: $\vDash A \leftrightarrow B$, (or, equivalently: "$A \vDash B$ and $A \vDash B$")

Logically Inconsistent: $\vDash \neg(A \wedge B)$

Valid$_{Semantic}$: $P_1, P_2, \ldots \vDash C$

Proof-Theoretic Concepts:

Theorem: $\vdash A$

Contradiction$_{Proof\text{-}Theoretic}$: $\vdash \neg A$

Provably Equivalent: $A \vdash B$ and $A \vdash B$

Provably Inconsistent: $\neg(A \wedge B) \vdash \bot$ (or, equivalently: "$\vdash \neg(A \wedge B)$")

Valid$_{Proof\text{-}Theoretic}$: $P_1, P_2, \ldots \vdash C$

Interesting question:

Can we understand these things as, ultimately, saying the same thing?

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NOTE: It is not a guarantee that these notions would line up. The proof theory and the semantics are completely detached from each other, and so we shouldn't expect this question to have an easy, simplistic answer.

TWO interesting questions:

Is Truth-Functional Logic *Sound*?

* Are all $\text{VALID}_{\text{Proof-Theoretic}}$ arguments also $\text{VALID}_{\text{Semantic}}$?

Is Truth-Functional Logic *Complete*?

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That is, are all the arguments for which we can give proofs also ones that can be shown to be valid using truth-tables?

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Is Truth-Functional Logic *Complete*?

* Are all VALID$_{\text{Semantic}}$ arguments also VALID$_{\text{Proof-Theoretic}}$?

That is, are all the arguments we can show to be Valid by checking every case using truth-tables also ones we can write down a proof for?

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How could we *show* this?

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Sketching the Proof

* We'll tackle this after the break!

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Core Rules of Natural Deduction for TFL

Rules that rely on previous lines in the *main* proof

Reit	Reiteration
$\wedge$**I**	Conjunction Introduction
$\wedge$**E**	Conjunction Elimination
$\vee$**I**	Disjunction Introduction
$\rightarrow$**E**	Conditional Elimination
$\leftrightarrow$**E**	Biconditional Elimination

Rules that *also* (generally) occur *within* or care *about* sub-proofs

$\rightarrow$**I**	Conditional Introduction
$\leftrightarrow$**I**	Biconditional Introduction
$\vee$**E**	Disjunction Elimination
$\neg$**E**	Negation Elimination
$\neg$**I**	Negation Introduction
IP	Indirect Proof
Ex	Explosion