Class 8 NATURAL DEDUCTION

Zee R. Perry

Let's do some logic

Natural Deduction

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is valid.

But it does this in a very different way than our truth-tables system!

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Natural deduction is a theory of ***formal*** **proofs**

Constructing a formal proof is a step-by-step process, and each step must be performed in accordance with the rules of a Natural Deduction system.

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Step A: Premises

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Step A: Premises

Step B: Apply Rules

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Step A: Premises

Step B: Apply Rules

Step C: Conclusion

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General Rules: (These apply at every step)

- **Numbered lines:** Every line in the proof is numbered.
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Step A: Premises - List all the premises.

- **Line:** List each premise on its own line.
- **Format:** Number each line you add. Draw the vertical stroke.
- **Right-Margin:** On each line with a premise, write “Pr.” on the far right-hand side of the proof.
- **At the end:** Draw a horizontal line under the final premise’s line.

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Example:

1	$A \wedge B$	Pr.
2	$B \to C$	Pr.

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How to Build a Proof in Three Easy Steps: **Step B**

Step B: Apply Rules - Add new lines according to rules.

- **Line:** Use a rule to determine what new lines you are allowed to add, given what's come before.
- **Format:** Number each line you add. Draw the vertical stroke. ***In addition,*** include any other formatting as required by the rules applied in this line or previous ones.
- **Right-Margin:** Write the rule that you used to generate that line, and the numbers of the lines the rule applied to.

How to Build a Proof in Three Easy Steps: **Step B**

Example:

1	$A \wedge (B \vee C)$	Pr.
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How to Build a Proof in Three Easy Steps: **Step C**

Step C: Conclusion - Step B ends when you reach the conclusion

- **Line:** If, during Step B, you manage to write down the conclusion, still following the rules of Natural Deduction, then you're done! ***Congrats!!***
- **Format:** Number each line you add. Draw the vertical stroke.
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How to Build a Proof in Three Easy Steps: **Step C**

Example:

1	$(A \vee C) \wedge B$	Pr.
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(***Psst!*** **– Our desired conclusion is 'C' !)**

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4	B	$\wedge$E, 1
5	C	$\rightarrow$E, 2, 4

(***Psst!*** **– Our desired conclusion is ‘*C*’ !)**

Step C: Conclusion - Step B ends when you reach the conclusion

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Proofs: It's *all* about Step B!

The Natural Deduction system has a large collection of rules which we follow to construct proofs, and they are what allow us to use Step B to get from the argument's premises to its conclusion.

Rule: Reiteration (“Reit.” or “R”)

The idea is this: The following argument-form is always valid, no matter what sentence you use.

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$$\frac{1.\ (P \leftrightarrow Q)}{\therefore\ (P \leftrightarrow Q)}$$

Rule: Reiteration ("Reit." or "R")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

$$\begin{array}{ll} 1. & (A \rightarrow (B \vee (C \wedge Z))) \\ \hline \therefore & (A \rightarrow (B \vee (C \wedge Z))) \end{array}$$

Rule: Reiteration (“Reit.” or “R”)

The idea is this: The following argument-form is always valid, no matter what sentence you use.

$$\begin{array}{l} 1.\ \mathscr{A} \\ \hline \therefore\ \mathscr{A} \end{array}$$

Reiteration Rule, “Reit.”

⋮		
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{A}$	Reit., n

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1. P
2. Q

$\therefore\ P \wedge Q$

Rule: Conjunction Introduction ("$\wedge$I" or "$\wedge$In")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. S
2. A

$\therefore\ S \wedge A$

Rule: Conjunction Introduction ("∧I" or "∧In")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. $(F \vee D)$
2. $(P \rightarrow \neg D)$

$\therefore\ (F \vee D) \wedge (P \rightarrow \neg D)$

Rule: Conjunction Introduction ("∧I" or "∧In")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. $(A \rightarrow (B \vee (C \wedge Z)))$
2. $\neg B$

$\therefore\ (A \rightarrow (B \vee (C \wedge Z))) \wedge \neg B$

Rule: Conjunction Introduction ("$\wedge$I" or "$\wedge$In")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. $\mathscr{A}$
2. $\mathscr{B}$

$\therefore\ \mathscr{A} \wedge \mathscr{B}$

Conjunction Introduction, "$\wedge$I"

	$\vdots$		
n		$\mathscr{A}$	(Some Rule) , (Line Numbers)
$\vdots$		$\vdots$	$\vdots$
m		$\mathscr{B}$	(Some Rule) , (Line Numbers)
$\vdots$		$\vdots$	$\vdots$
u		$\mathscr{A} \wedge \mathscr{B}$	$\wedge$I, n, m

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$$\begin{array}{l} 1.\ (Q \wedge R) \\ \hline \therefore\ Q \end{array} \qquad \begin{array}{l} 1.\ (Q \wedge R) \\ \hline \therefore\ (R) \end{array}$$

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$$\begin{array}{l} 1.\ (P \to \neg D) \wedge (F \leftrightarrow D) \\ \hline \therefore\ P \to \neg D \end{array}$$

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$\therefore\ \neg B$

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Conjunction Elimination, "∧E"

	⋮	
n	$\mathscr{A} \wedge \mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
u	$\mathscr{A}$	∧E, n
⋮	⋮	⋮
v	$\mathscr{B}$	∧E, n