Class 9 Proof Theory, continued

Zee R. Perry

Let's do some logic

Correcting an Incorrect "Proof"

1	$(\neg B \wedge C) \rightarrow D$	Pr.
2	$(C \wedge \neg A) \leftrightarrow (\neg A \wedge \neg B)$	
3	$\neg A \wedge C$	Pr.
4	C	$\wedge$E, 3
5	$\neg A$	$\wedge$E, 1
6	$C \wedge \neg A$	$\wedge$E, 4, 5
7	$\neg A \wedge \neg B$	$\leftrightarrow$E, 2, 6
8	$\neg B$	$\wedge$E, 7
9	$\neg B \wedge C$	$\wedge$I, 4, 8

10	D	Reit., 11
11	D	$\rightarrow$E, 9
12	$D \wedge D$	$\wedge$I, 10
WTS:	$(D \wedge D) \vee E$	
13	$(D \wedge D) \vee E$	$\vee$I, 12

Second Incorrect "Proof":

1	$(P \vee Q) \wedge S$	Pr.
2	$P \wedge \neg R$	Pr.
Want to Show:	$(\neg R \wedge S) \wedge (S \wedge \neg R)$	Concl.
3	S	Reit., 1
4	$P \vee Q$	$\wedge$E, 1
5	$P \wedge \neg R$	Reit., 2
6	$\neg R$	$\wedge$E, 5
7	$\neg R \wedge S$	$\wedge$I, 3, 6
8	$S \wedge \neg R$	$\wedge$I, 7
9	$(\neg R \wedge S) \wedge (S \wedge \neg R)$	$\wedge$I, 3, 6, 7, 8
10	$(\neg R \wedge S) \wedge (S \wedge \neg R)$	Reit., 9

Rule: Reiteration ("Reit." or "R")

The idea is this: The following argument-form is always valid, no matter what sentence you use.

1. P	**1.** $(A \leftrightarrow G)$	**1.** I'm busy
$\therefore$ P	$\therefore$ $(A \leftrightarrow G)$	$\therefore$ I'm busy

Rule: Reiteration (“Reit.” or “R”)

The idea is this: The following argument-form is always valid, no matter what sentence you use.

Example 1	Example 2	Example 3
1. P	**1.** $(A \leftrightarrow G)$	**1.** I’m busy
$\therefore\ P$	$\therefore\ (A \leftrightarrow G)$	$\therefore$ I’m busy

Reiteration Rule, “Reit.”

Line		Sentence	Justification
		⋮	
n		$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮		⋮	⋮
m		$\mathscr{A}$	Reit., n

Rule: Conjunction Introduction ("∧I" or "∧In")

1. P
2. $Q \vee P$

$\therefore\ P \wedge (Q \vee P)$

1. $(A \leftrightarrow G)$
2. $\neg S$

$\therefore\ (A \leftrightarrow G) \wedge \neg S$

1. There's a giraffe in this building
2. Rutgers is in New Jersey

∴ There's a giraffe in this building *and* Rutgers is in NJ

Rule: Conjunction Introduction ("$\wedge$I" or "$\wedge$In")

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Conjunction Introduction, "$\wedge$I"

n	$\mathscr{A}$	(Some Rule) , (Line Numbers)	
⋮	⋮	⋮	
m	$\mathscr{B}$	(Some Rule) , (Line Numbers)	
⋮	⋮	⋮	
u	$\mathscr{A} \wedge \mathscr{B}$	$\wedge$I ,	n, m

Rule: Conjunction Elimination ("$\wedge$E" or "$\wedge$Out")

1. $P \wedge (Q \vee P)$

$\therefore\ Q \vee P$

1. $\neg S \wedge (A \leftrightarrow G)$

$\therefore\ \neg S$

1. There's a giraffe in this building *and* Rutgers is in NJ

$\therefore$ There's a giraffe in this building

Rule: Conjunction Elimination ("∧E" or "∧Out")

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1. There's a giraffe in this building *and* Rutgers is in NJ

∴ There's a giraffe in this building

Conjunction Elimination, "∧E"

n	$\mathscr{A} \wedge \mathscr{B}$	(Some Rule)	,	(Line Numbers)
⋮	⋮	⋮		
u	$\mathscr{A}$	∧E	,	n
⋮	⋮	⋮		
v	$\mathscr{B}$	∧E	,	n

Rule: Disjunction Introduction ("$\vee$I" or "$\vee$In")

Okay, hear me out. This is gonna seem **WILD**. Disjunction introduction is the reasoning involved in the following arguments (all of which are **valid!**):

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1. It's Wednesday.

$\therefore$ Either it's Wednesday or I'm a kettle of fish.

1. $\neg S \wedge A$

$\therefore$ $(\neg S \wedge A) \vee B$

1. P

$\therefore$ $P \vee Q$

Rule: Disjunction Introduction ("$\vee$I" or "$\vee$In")

1. It's Wednesday.

∴ Either it's Wednesday or I'm a kettle of fish.

1. $\neg S \wedge A$

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1. P

∴ $P \vee Q$

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1. It’s Wednesday.

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1. $\neg S \wedge A$

∴ $(\neg S \wedge A) \vee B$

1. P

∴ $P \vee Q$

Disjunction Introduction, “∨I”

For **ANY** “$\mathscr{B}$” !

n	$\mathscr{A}$	(Some Rule) , (Line Numbers)	
⋮	⋮	⋮	
u	$\mathscr{A} \vee \mathscr{B}$	∨I	, n

Rule: Conditional Elimination ("→E" or "→Out")

Conditional Elimination works exactly as you'd expect it would. If you have a conditional on some line, and the antecedent of that conditional on another line, then you can write down the consequent of that conditional.

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1. If it's raining then my hair will get wet.
2. It's raining.
$\therefore$ My hair will get wet

1. $(\neg S \wedge A) \rightarrow (B \leftrightarrow A)$
2. $\neg S \wedge A$
$\therefore$ $B \leftrightarrow A$

1. $P \rightarrow Q$
2. P
$\therefore$ Q

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2. P

$\therefore\ Q$

Conditional Elimination, "→E"

n	$\mathscr{A} \rightarrow \mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
u	$\mathscr{B}$	→E , n,m

Proof without Rules & Line Numbers

1	$P \wedge S$	____, ____
2	$S \to R$	____, ____
WTS:	$R \vee E$	
3	P	____, ____
4	S	____, ____
5	R	____, ____
6	$R \vee E$	____, ____

Proof without TFL Sentences after Premises.

1	$(Q \wedge P) \to S$	Pr.
2	$P \wedge Q$	Pr.
WTS:	$S \wedge Q$	
3	________	$\wedge$E, 2
4	________	$\wedge$E, 2
5	________	$\wedge$I, 3, 4
6	________	$\to$E, 1, 5
7	________	$\wedge$I, 4, 6

New Rules: Adding "Layers" to our proofs!

The remaining rules in our Natural Deduction system for TFL will involves **Assumptions** and **Sub-proofs**.

These rules will indicate when you are allowed to start a *new* proof **within** your current proof.

The simplest form of this comes from Conditional Introduction ("$\rightarrow$I").

Rule: Conditional Introduction ("→I" or "→In")

Conditional Introduction is a rule that lets you write down a **conditional** statement (i.e. a sentence whose main connective is '→'), given some lines earlier in your proof. What would this even look like? What—*IN GENERAL*—is required for "$P \rightarrow Q$" to be true?

Rule: Conditional Introduction ("→I" or "→In")

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First Pass: Surely, **IF** "P" is true, **THEN** "Q" must also be true.

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What—*IN GENERAL*—is required for "$P \rightarrow Q$" to be true?

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But it also *doesn't require P to actually be true*! So how could we make sense of this in our proof system?

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Observation: Actually, this is kind of familiar... You know what's **also** like that? **VALIDITY!**

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But how could we—while in the middle of a proof!—show that some *other* argument is valid??

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But how could we—while in the middle of a proof!—show that some *other* argument is valid??

Simplest Solution: Put a proof on your proof so you can prove while you prove! So how the heck would *that* work?!

Sub-proofs and Assumptions

Within a proof, you can start **another** proof once you've moved past the premises stage:

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WTS: $A \rightarrow B$

Sub-proofs and Assumptions

When you start a sub-proof, you introduce ***Assumptions*** instead of premises.

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Any line you add *within* your sub-proof can reference any line above it, ***including lines from your main proof!***

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	$\quad A$	Ass.
4	$\quad C$	$\to$E, 1, 3

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2	$C \to B$	Pr.
WTS:	$A \to B$	
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4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4

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1	$A \rightarrow C$	Pr.	
2	$C \rightarrow B$	Pr.	
WTS:	$A \rightarrow B$		
3	A	Ass.	
4	C	$\rightarrow$E, 1, 3	
5	B	$\rightarrow$E, 2, 4	
6			

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

Sub-proofs and Assumptions

1	$A \rightarrow C$	Pr.
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WTS:	$A \rightarrow B$	
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4	C	$\rightarrow$E, 1, 3
5	B	$\rightarrow$E, 2, 4
6	B	Reit, 5

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WTS:	$A \to B$	
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4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4
6		

But it doesn't work in reverse! You *cannot* reference any line from the sub-proof after you've exited and returned to the main proof!

However, there are some rules that can only be applied *if a sub-proof has occurred earlier in the proof*! One such rule is $\to$I

Sub-proofs and Assumptions

1	$A \to C$	Pr.
2	$C \to B$	Pr.
WTS:	$A \to B$	
3	A	Ass.
4	C	$\to$E, 1, 3
5	B	$\to$E, 2, 4
6		$\to$I, 3–5

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5	B	$\to$E, 2, 4
6	$A \to B$	$\to$I, 3–5

However, there are some rules that can only be applied *if a sub-proof has occurred earlier in the proof*! One such rule is $\to$I

Rule: Conditional Introduction (“$\to$I” or “$\to$In”)

Conditional Introduction appeals to the following kind of reasoning:

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Since $\dfrac{\textbf{1.}\ \ P}{\therefore\ \ Q}$ is a valid argument, it must be true that $P \rightarrow Q$

If we assume that it'll rain tonight, then it follows that I'll get my hair wet. So, *no matter what* the weather's like, we know that the conditional "If it rains, then I'll get my hair wet" is true.

Rule: Conditional Introduction ("→I" or "→In")

Conditional Introduction, "→I"

⋮	⋮	⋮
n	$\mathscr{A}$	Ass.
⋮	⋮	⋮
u	$\mathscr{B}$	(Some Rule) , (Line Numbers)
1		
⋮	⋮	⋮
v	$\mathscr{A} \rightarrow \mathscr{B}$	→I , n–m

Rule: Disjunction Elimination (“$\vee$E” or “$\vee$Out”)

Disjunction Elimination *also* relies on the use of sub-proofs!

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Disjunction Elimination *also* relies on the use of sub-proofs!

1. X is either in the library or in the student center.

2. If X is in the library, then X is on College Ave.

3. If X is in the student center, then X is on College Ave.

$\therefore$ X is on College Ave!

Rule: Disjunction Elimination ("∨E" or "∨Out")

Disjunction Elimination, "∨E"

i	$\mathscr{A} \lor \mathscr{B}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
n	$\vert\ \mathscr{A}$	Ass.
m	$\vert\ \mathscr{C}$	(Some Rule) , (Line Numbers)
⋮		⋮
u	$\vert\ \mathscr{B}$	Ass.
v	$\vert\ \mathscr{C}$	(Some Rule) , (Line Numbers)
j	$\mathscr{C}$	∨E , i, n–m, u–v

Rule: Negation Elimination ("¬E" or "𝕏In" or "⊥In")

Negation Elimination is also sometimes known as Contradiction Introduction, because it only "eliminates" negations in a special case, when its the negation of a sentence that appears elsewhere in your proof.

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Negation Elimination, "¬E"		
⋮	⋮	⋮
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\neg\mathscr{A}$	(Some Rule) , (Line Numbers)
j	$\bot$	¬E , n, m

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⋮	⋮	⋮
n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
⋮	⋮	⋮
m	$\neg\mathscr{A}$	(Some Rule) , (Line Numbers)
j	$\mathbb{X}$	¬E , n, m

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n	$\mathscr{A}$	(Some Rule) , (Line Numbers)
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Rule: Negation Introduction ("¬I" or "¬In")

Negation introduction is a way of introducing a negation by "eliminating" a contradiction (when that contradiction arises **IN A SUB-PROOF!**). Specifically, if a sub-proof leads you to a contradiction, then you ***know*** that the assumptions you made *cannot* be true!

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Negation Introduction, “¬I”

⋮	⋮	⋮
n	$\mathscr{A}$	Ass.
⋮	⋮	⋮
m	$\bot$	(Some Rule) , (Line Numbers)
j	$\neg\mathscr{A}$	¬I , n–m

Rule: Indirect Proof ("IP")

Indirect Proof, also called "Proof by Contradiction", is extremely similar to negation introduction. However, instead of *introducing* a negation at the end, we're *assuming* a negated statement, in order to show that this leads to a contradiction. If assuming that "P" is false leads to a contradiction, then it must* be that P is true!

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Indirect Proof, "IP"

⋮	⋮	⋮
n	$\neg\mathscr{A}$	Ass.
⋮	⋮	⋮
m	$\bot$	(Some Rule) , (Line Numbers)
j	$\mathscr{A}$	IP , n–m

Rule: Explosion (“Ex” or “⊥E” or “XXOut”)

The Explosion rule has a fun name because it describes a really wild feature of TFL: Literally anything follows from a contradiction. That is, any argument with a contradiction among its premises is valid, no matter what else is in the argument!

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Explosion, "Ex"		
n	$\bot$	(Some Rule) , (Line Numbers)
$\vdots$	$\vdots$	$\vdots$
m	$\mathscr{A}$	Ex , n

Homework!

From Chapter 15

Block A: Correct both Proofs

Block B: Complete both Proofs

Block C: Do ONLY 1, 2, 3, 4