Class 7:
Wrap up: TFL Semantics
Start on: Natural Deduction

LOGIC

WITH ZEE PERRY

Admin: Midterm Quiz 1

Midterm Quiz 1 is going to become available later this week, it will be a 4-hour timed take-home administered through Brightspace!

That's right, we have a Brightspace page now!

Once I publish the page, it'll consist mostly of just the Midterm and a link to our regular course webpage. Later, I will probably migrate quizlets and things over there as well.

However, the main source of info for the course will continue to be the course website (but having a Brightspace page will make things much easier!)

Admin: Office Hours

I will be holding online Office Hours
and Study Hall sessions during the following timeframes:

Office Hours: (all in Shanghai time)

- Thursday Evening 8:00pm to 10:00pm (online)
- Saturday Morning 8:30am to 11:30am (online)

The office hours will be at the same Zoom link as our regular class

Truth-Functional Logic (TFL)

	Sentence Component	Representation in English (a popular Natural Language)	Representation in TFL
SENTENCES	Atomic Sentences	"I love to eat pizza" "Electrons are point-particles" (like, could be *literally any sentence*)	*A, B, C, D, E, F, G, H, …*
SENTENTIAL CONNECTIVES	Negation	"not..", "it's not the case that…", "it's not true that…"	¬ (Alternatively, '~')
	Conjunction	".. and..", ".. but..", ".. however .."	∧ (Alternatively, '&', '.')
	Disjunction	"Either… or..", ".. or..",	∨
	Material Conditional	"if.. then.." , ".. only if .." (or ".. if ..", but only in reverse!)	→, (Alternatively, '⊃')
	Bi-Conditional	".. if and only if..", ".. iff..", "… just in case..",	↔ (Alternatively,

Important: Every complex sentence has a **Main Connective!**

Atomic sentences contain no connectives at all.

But any complex sentence in TFL is made using one or more connectives. A sentence's **main connective** is the one that determines the truth-value of the **whole sentence**.

You can identify the **main connective** of a sentence by looking for the connective which connects to the **entire rest of the sentence**, rather than just part of it.

Why (internal) **parentheses** are **important!!**

Parentheses are important because they help you figure out what the **main connective** of a sentence is!

- (by showing you which sentences are hooked on to which connective.)

Quizlet Q1: Circle the **main connective** of these three sentences

$\neg((A \rightarrow B) \wedge (C \vee D))$ $((\neg A \rightarrow B) \wedge (C \vee D))$ $((\neg A \rightarrow (B \wedge C)) \vee D)$

QUIZLET PAGE: **tinyurl.com/Quizlet-922**

Semantic Notion: **Tautologies**

Recall: in TFL, a "possible case" is just the assignment of Truth-Values to each of the ***atomic*** sentences.

- We'll use the book's term, "**Valuation**" or "**Possible Valuation**"

Sentence "$\mathcal{A}$" is a **Tautology** iff it is true <u>in every case</u>

Sentence "$\mathcal{A}$" is a **Tautology** iff it is true <u>on every valuation</u>

The sentence on the right is one of the simplest tautologies.

$(P \lor \neg P)$			
P	$\lor$	$\neg$	P
T	T	F	T
F	T	T	F

Semantic Notion: **Tautologies**

TAUTOLOGY = Sentence is ALWAYS true

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P	$\vee$	$\neg$	P
T	T	F	T
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(A → A)		
A	→	A
T	T	T
F	T	F

Semantic Notion:
Tautologies

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$(A \rightarrow A)$		
A	$\rightarrow$	A
T	T	T
F	T	F

Check if this sentence is a tautology: YES

$(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	(A	↔	B)	↔	¬	(A	↔	¬	B)
T	T	T	T	T	T	T	T	F	F	T
F	T	F	F	T	T	F	F	T	F	T
T	F	T	F	F	T	F	T	T	T	F
F	F	F	T	F	T	T	F	F	T	F

Check if this sentence is a tautology: YES

$$(A \rightarrow B) \vee (B \rightarrow A)$$

A	B	(A	→	B)	v	(B	→	A)
T	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	F	F
T	F	T	F	F	T	F	T	T
F	F	F	T	F	T	F	T	F

Semantic Notion: **Contradictions**

CONTRADICTION = Sentence is ALWAYS false

Contradictions are the opposite of a Tautology. Rather than always being true, contradictions are always false.

"$\mathscr{A}$" is a **Contradiction** iff it is false <u>in every case</u>

"$\mathscr{A}$" is a **Contradiction** iff it is false <u>on every valuation</u>

$(Q \wedge \neg Q)$			
Q	$\wedge$	$\neg$	Q
T	F	F	T
F	F	T	F

The sentence on the right is one of the simplest contradictions

It's very similar to the simple tautology, you just switch out the 'or' ('V') for the 'and' ('Λ').

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It's very similar to the simple tautology, you just switch out the 'or' ('∨') for the 'and' ('∧').

$(Q \wedge \neg Q)$			
Q	$\wedge$	$\neg$	Q
T	F	F	T
F	F	T	F

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Q	$\wedge$	$\neg$	Q
T	F	F	T
F	F	T	F

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CONTRADICTION = Sentence is ALWAYS false

Not a Contra

$C \to \neg C$			
C	→	¬	C
T	F	F	T
F	T	T	F

Semantic Notion: (TF)-Logical **Equivalence**

Logically Equivalent =
The sentences ***ALWAYS*** have same truth-value as each other

We had homework questions about logical equivalence in ***natural*** language. With truth-tables, we can understand logical equivalence in the formal language, TFL.

"$\mathscr{A}$" and "$\mathscr{B}$" are **Equivalent** iff they have the *same* truth-value <u>on every valuation</u>.

"$\mathscr{A}$" and "$\mathscr{B}$" are **Equivalent** iff the sentence "$(\mathscr{A} \leftrightarrow \mathscr{B})$" is a ***tautology***.

$((\neg R \wedge \neg S) \leftrightarrow \neg (R \vee S))$

R	S	$(\neg$	R	$\wedge$	$\neg$	$S)$	$\leftrightarrow$	$\neg$	$(R$	$\vee$	$S)$
T	T	F		F	F		T	F	T	T	T
F	T	T		F	F		T	F	F	T	T
T	F	F		F	T		T	F	T	T	F
F	F	T		T	T		T	T	F	F	F

Semantic Notion: Jointly **Satisfiable**

We also had homework questions about when pairs of sentences in ***natural*** language were "Jointly possible" or "jointly impossible". With the tools of TFL and truth-tables, we can now understand a very similar semantic notion: **joint satisfiability**.

"$\mathcal{A}_1$" "$\mathcal{A}_2$",..., "$\mathcal{A}_n$" are **Jointly Satisfiable** iff
there exists *some* valuation which makes them all true.

"$\mathcal{A}_1$" "$\mathcal{A}_2$",..., "$\mathcal{A}_n$" are **Jointly Unsatisfiable** iff
there *doesn't* exist a valuation which makes them all true.

$(A \wedge (A \vee B))$

A	$\wedge$	$(A$	$\vee$	$B)$
T	T	T	T	T
F	F	F	T	T
T	T	T	T	F
F	F	F	F	F

Check if this sentence is a tautology:

$(D \land \neg D) \rightarrow G$

D	G	(D	∧	¬	D)	→	G
T	T	T	F	F	T	T	T
F	T	F	F	T	F	T	T
T	F	T	F	F	T	T	F
F	F	F	F	T	F	T	F

Check if this sentence is a tautology:

(D ∧ ¬D) → G							
D	G	(D	∧	¬	D)	→	G
T	T	T	F	F	T	T	T
F	T	F	F	T	F	T	T
T	F	T	F	F	T	T	F
F	F	F	F	T	F	T	F

Check this sentence's main connective:

(¬P v ¬M) ↔ M								
P	M	(¬	P	v	¬	M)	↔	M
T	T	F	T	F	F	T	F	T
F	T	T	F	T	F	T	T	T
T	F	F	T	T	T	F	F	F
F	F	T	F	T	T	F	F	F

Check this sentence's main connective:

(¬P v ¬M) ↔ M								
P	M	(¬	P	v	¬	M)	↔	M
T	T	F	T	F	F	T	F	T
F	T	T	F	T	F	T	T	T
T	F	F	T	T	T	F	F	F
F	F	T	F	T	T	F	F	F

Check this sentence's main connective:

¬¬(¬A ∧ ¬B)								
A	B	¬	¬	(¬	A	∧	¬	B)
T	T	F	T	F	T	F	F	T
F	T	F	T	T	F	F	F	T
T	F	F	T	F	T	F	T	F
F	F	T	F	T	F	T	T	F

Check this sentence's main connective:

¬¬(¬A ∧ ¬B)

A	B	¬	¬	(¬	A	∧	¬	B)
T	T	F	T	F	T	F	F	T
F	T	F	T	T	F	F	F	T
T	F	F	T	F	T	F	T	F
F	F	T	F	T	F	T	T	F

Finally, we can get *precise* about: **Entailment** and **Validity**

"$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" **entail** "$\mathcal{B}$", iff this argument is **valid**:

1. $\mathcal{A}_1$
2. $\mathcal{A}_2$

⋮

n. $\mathcal{A}_n$

---

c. $\mathcal{B}$

Put Another Way…

The sentences, "$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" **entail** (in TFL) the sentence "$\mathcal{B}$" iff there is NO single valuation that:

makes "$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" all true

and also makes "$\mathcal{B}$" false.

Aside: Let's introduce a New SYMBOL for an OLD language (specifically, English)

Double turnstile: ' $\vDash$ '

- So-called because it looks like a turnstile you'd see on a subway platform, an amusement park, a stadium, etc.
- Except that it's "double" because it's got ***two*** little bars sticking out the side instead of one.

The double turnstile is *NOT* a symbol in TFL!!

It's a symbol of *English*!! That is, we've **added it** to English in order to make it easier for **us** to **talk about logic** in a natural language.

Aside:

The Double Turnstile, ‘ $\vDash$ ’

The double turnstile is ***NOT*** a symbol in TFL!!

- It’s a symbol of *English* (or “modified English”)! To make it easier for **us** to **talk about logic** in a natural language.

The double-turnstile ***IS*** a simple way for us to talk about ***entailment***!

So, for instance:

$$\text{“ } (P \vee Q),\ \neg P \ \vDash \ Q \text{ ”}$$

Stands for:

- “ The sentences ‘$(P \vee Q)$,’ and ‘$\neg P$’ **entail** ‘Q’. ”

aka:

- “ The argument with premises: ‘$(P \vee Q)$,’ and ‘$\neg P$’ and conclusion: ‘Q’ is **valid**. ”

Finally, we can get *precise* about: **Entailment** and **Validity**

"$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" **<u>entail</u>** "$\mathcal{B}$", iff this argument is **<u>valid</u>**:

1. $\mathcal{A}_1$
2. $\mathcal{A}_2$

⋮

n. $\mathcal{A}_n$

---

c. $\mathcal{B}$

Put Another Way…

The sentences, "$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" <u>entail</u> (in TFL) the sentence "$\mathcal{B}$" iff there is NO single valuation that:

makes "$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" all true

and also makes "$\mathcal{B}$" false.

Finally, we can get *precise* about: **Entailment** and **Validity**

$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{B}$ iff this argument is **valid**:

1. $\mathcal{A}_1$
2. $\mathcal{A}_2$

⋮

n. $\mathcal{A}_n$

---

c. $\mathcal{B}$

Put Another Way…

$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{B}$

iff there is NO single valuation that:

- makes "$\mathcal{A}_1$" "$\mathcal{A}_2$",…, "$\mathcal{A}_n$" all true
- and also makes "$\mathcal{B}$" false.

Let's test if :

$\neg P, (P \rightarrow Q) \vDash Q$

Let's test if this argument is valid or invalid:

1. $\neg P$
2. $(P \rightarrow Q)$

---

3. Q

(premises)

$\neg P$		$(P \rightarrow Q)$			Q
$\neg$	P	P	$\rightarrow$	Q	Q
F	T	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	F
T	F	F	T	F	F

(conclusion)

Counter example!!!

Let's test if : TRUE

$\neg B, (B \vee A) \vDash A$

Let's test if this argument is valid or invalid:

1. $\neg B$
2. $(B \vee A)$

---

3. A

VALID

(premises)

$\neg B$		$(B \vee A)$			A (conclusion)
¬	B	B	v	A	A
F	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
T	F	F	F	F	F

But does this really help us **understand** validity?

We've got a precise way to enumerate every possibility in TFL, so we can easily ***check*** if an argument is such that: "there's no possibility where all the premises are true and conclusion false".

But we wanted to **understand** validity on a deeper lever. Specifically, we want to understand *why* certain arguments are valid – i.e. what ***kinds of reasoning*** produce valid arguments.

To achieve this, we'll introduce a system to investigate arguments in TFL, that **doesn't** rely on truth-tables.

A Natural Deduction system

Natural Deduction System

A Natural Deduction system is a system for **evaluating arguments** (i.e. determining if they're valid or invalid) in TFL

It requires significantly less "busy-work" than truth-table methods, because it relies on a very different core concept.

Natural Deduction vs Truth-Tables

Natural deduction requires less "busy-work" than truth-tables when checking if an argument is valid. For example, is this argument valid?

1. $(P_1 \rightarrow Q_1)$

$\therefore \quad (P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5) \rightarrow (Q_1 \vee Q_2 \vee Q_3 \vee Q_4 \vee Q_5)$

To check with truth-tables, you'd need 1,024 rows!!

But I can convince you its valid pretty easily!

Natural Deduction vs Truth-Tables

Here's a valid argument:

1. If it's raining, then I'll get my hair wet.

∴ If (it's raining and it's a Monday), then (either I'll get my hair wet or I'll find a penny).

And you'd get another valid argument if you added another conjunct to the antecedent ("if"-clause), or another disjunct to the consequent ("then"-clause), of the conclusion.

1. If it's raining, then I'll get my hair wet.

∴ If (it's raining and it's a Monday and I'm tired), then (either I'll get my hair wet or I'll find a penny or I'll grow wings).

Natural Deduction vs Truth-Tables

Hence, this argument **IS** valid!

1. $(P_1 \rightarrow Q_1)$

$\therefore \ (P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5) \rightarrow (Q_1 \vee Q_2 \vee Q_3 \vee Q_4 \vee Q_5)$

And, by the same reasoning, so is **THIS** argument:

1. $(P_1 \rightarrow Q_1)$

$\therefore \ (P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \wedge P_8)$
$\rightarrow (Q_1 \vee Q_2 \vee Q_3 \vee Q_4 \vee Q_5 \vee Q_6 \vee Q_7 \vee Q_8)$

But you'd need a truth-table with $2^{16} = 65{,}536$ rows to show that it's valid using the truth-table method!

Natural Deduction consists of rules

Instead of checking every possible distribution of truth-values, Natural Deduction relies on **rules**.

These rules are super-simple argument-forms that we know are valid **no matter what** sentences you plug into them.

And, so, if we can show that you could apply a **series** of these rules to get you from the premises of an argument to the conclusion, then we'll have **proved** that the argument is valid.