

Class 12 - Soundness and Completeness of TFL

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Let's do some logic!

$$A \rightarrow B, A \rightarrow C \vdash A \rightarrow (B \wedge C)$$

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1 $A \rightarrow B$ Pr.

2 $A \rightarrow C$ Pr.

WTS: $A \rightarrow (B \wedge C)$

3

??

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$A \rightarrow (B \wedge C)$ \rightarrow I, ??-??

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<hr/>			
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4			
		B	$\rightarrow E, 1, 3$
		$B \wedge C$?? $\wedge I,$
			??
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5	C	\rightarrow E, 2, 3
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6			$B \wedge C$ \wedge I, 4, 5
			??
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6			$B \wedge C$ $\wedge I, 4, 5$
			??
		$A \rightarrow (B \wedge C)$	$\rightarrow I, 3-6$

$$\neg A \rightarrow \neg B \vdash B \rightarrow A$$

Filling-in Citations Proofs

1	$W \rightarrow \neg B$	Pr.
2	$A \wedge W$	Pr.
3	$B \vee (J \wedge K)$	Pr.
4	W	
5	$\neg B$	
6	$\neg B$	
7	$J \wedge K$	
8	K	

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1	$L \leftrightarrow \neg O$	Pr.
2	$L \vee \neg O$	Pr.
3	$\neg L$	
4	$\neg O$	
5	L	
6	\perp	
7	$\neg\neg L$	
8	L	

$$E \vee F, F \vee G, \neg F \therefore E \wedge G$$

Any questions about this one? If not I'll move on.

Semantics and Proof-theory Collide!

We've covered *multiple* definitions of very similar concepts:

Remember this semantic concept?

Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: “ \vDash ”

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Reminder: Meaning of the Double Turnstile “ \vDash ”

Instead of saying:

- “The sentences ‘ \mathcal{A} ’, ‘ \mathcal{B} ’, and ‘ \mathcal{C} ’ together ENTAIL the sentence ‘ \mathcal{P} ’.”
- “There’s NO POSSIBLE CASE where all the sentences ‘ \mathcal{A} ’, ‘ \mathcal{B} ’, and ‘ \mathcal{C} ’, are true and the sentence ‘ \mathcal{P} ’ is false.”

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- “There’s NO POSSIBLE CASE where all the sentences ‘ A ’, ‘ B ’, and ‘ C ’, are true and the sentence ‘ P ’ is false.”

We can say: “ $A, B, C \vDash P$ ”

Some Proof-theoretic concepts!

We can now introduce the **Single Turnstile**, “ \vdash ”! It is specifically focused on **what can be proved** within our system:

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- "From premises ' \mathcal{A} ', ' \mathcal{B} ', and ' \mathcal{C} ' you can **PROVE** the sentence ' \mathcal{P} '."

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- "From premises ' \mathcal{A} ', ' \mathcal{B} ', and ' \mathcal{C} ' you can **PROVE** the sentence ' \mathcal{P} '."
- "There exists at least one **PROOF** that (1) follows all the formal rules, (2) has premises ' \mathcal{A} ', ' \mathcal{B} ', and ' \mathcal{C} ', and (3) has, **as its final line**, the sentence ' \mathcal{P} '."

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We can say: " $\mathcal{A}, \mathcal{B}, \mathcal{C} \vdash \mathcal{P}$ "

Some Proof-theoretic concepts!

The Double Turnstile, “ \vDash ”, is about
the existence of VALUATIONS.

I.e. it's about what sentences can be
true and false at the same time (in,
e.g., a given row of a truth-table).

The Single Turnstile, “ \vdash ”, is
about *the existence of PROOFS.*

I.e. it's about whether there's a
way to construct a formal proof
that has those sentences located
at specific lines.

Things we can define using ' \vdash '

When a sentence is a "THEOREM of TFL"

To say that "Sentence ' \mathcal{A} ' is a THEOREM of TFL" is just to say:

Things we can define using '⊢'

When a sentence is a "THEOREM of TFL"

To say that "Sentence 'A' is a THEOREM of TFL" is just to say:

$$\vdash A$$

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To say that "Sentence ' \mathcal{A} ' is a THEOREM of TFL" is just to say:

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This means that \mathcal{A} can be proved using *no* premises at all!

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$$\mathcal{A} \vdash \mathcal{B} \quad \text{and} \quad \mathcal{B} \vdash \mathcal{A}$$

This means that \mathcal{A} can be proved from \mathcal{B} , **and vice versa!**

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To say that "Sentences '*A*', '*B*', '*C*', etc. are PROVABLY INCONSISTENT"
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$$A, B, C, \dots \vdash \perp$$

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or

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or

$$A, B, C, \dots \not\vdash \perp$$

This means you **can't** prove a contradiction from '*A*', '*B*', '*C*', etc.

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row in that sentence's truth-table is a "F"

Logically Equivalent: These sentences always have the *same* truth-value for each row of their shared truth-table.

Logically Inconsistent: There's no row of their shared truth table where both sentences are true.

Valid_{Semantic}: In **ALL** the rows where every premise is assigned a "T", the conclusion is also assigned a "T".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction_{Proof-Theoretic}: You can prove the sentence's *negation* from no premises.

Provably Equivalent: You can prove one sentence using the other as your premise, *and vice versa*.

Provably Inconsistent: You can prove a contradiction using those sentences as two premises.

Valid_{Proof-Theoretic}: There exists a proof from those premises to that conclusion.

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row of that sentence's *negation's* truth table is "T".

Logically Equivalent: The biconditional between those two sentences is a *tautology*

Logically Inconsistent: The conjunction of those two sentences is a *contradiction*.

Valid_{Semantic}: There is **NO** row where all the premises are assigned "T" and the conclusion is assigned an "F".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction_{Proof-Theoretic}: You can prove the sentence's *negation* from no premises.

Provably Equivalent: You can prove one sentence using the other as your premise, *and vice versa*.

Provably Inconsistent: You can prove a contradiction using those sentences as two premises.

Valid_{Proof-Theoretic}: There exists a proof from those premises to that conclusion.

Semantic Concepts:

Tautology: $\models A$

Contradiction: $\models \neg A$

Logically Equivalent: $\models A \leftrightarrow B$,
(or, equivalently: " $A \models B$ and
 $A \models B$ ")

Logically Inconsistent: $\models \neg(A \wedge B)$

Valid_{Semantic}: $P_1, P_2, \dots \models C$

Proof-Theoretic Concepts:

Theorem: $\vdash A$

Contradiction_{Proof-Theoretic}: $\vdash \neg A$

Provably Equivalent: $A \vdash B$ and
 $A \vdash B$

Provably Inconsistent: $\neg(A \wedge B) \vdash$
 \perp (or, equivalently: " $\vdash \neg(A \wedge$
 $B)$ ")

Valid_{Proof-Theoretic}: $P_1, P_2, \dots \vdash C$

Interesting question:

Can we understand these things as, ultimately, saying the same thing?

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NOTE: It is not a guarantee that these notions would line up. The proof theory and the semantics are completely detached from each other, and so we shouldn't expect this question to have an easy, simplistic answer.

TWO interesting questions:

Is Truth-Functional Logic *Sound*?

- * Are all $\text{VALID}_{\text{Proof-Theoretic}}$ arguments also $\text{VALID}_{\text{Semantic}}$?

Is Truth-Functional Logic *Complete*?

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That is, are all the arguments for which we can give proofs also ones that can be shown to be valid using truth-tables?

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Is Truth-Functional Logic *Complete*?

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That is, are all the arguments we can show to be Valid by checking every case using truth-tables also ones we can write down a proof for?

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- * Are all sentences that are THEOREMS of TFL also TAUTOLOGIES?

That is, are all sentences that we can (from no premises) construct a formal proof of also a sentence whose truth-table has only "T"s in every row under the main connective?

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Proof Sketch: TFL is *Sound*

We want to show that: “If $\mathcal{P} \vdash \mathcal{C}$, then $\mathcal{P} \models \mathcal{C}$ ”

- * We show this via an *informal* proof, since we’re making a proof *about* logic, but not doing logic (we’re doing *meta*-logic!)

Core Rules of Natural Deduction for TFL

Rules that rely on previous lines
in the *main* proof

Reit	Reiteration
\wedge I	Conjunction Introduction
\wedge E	Conjunction Elimination
\vee I	Disjunction Introduction
\rightarrow E	Conditional Elimination
\leftrightarrow E	Biconditional Elimination

Rules that *also* (generally) occur
within or *care about* sub-proofs

\rightarrow I	Conditional Introduction
\leftrightarrow I	Biconditional Introduction
\vee E	Disjunction Elimination
\neg E	Negation Elimination
\neg I	Negation Introduction
IP	Indirect Proof
Ex	Explosion