Class 12 - Soundness and Completeness of TFL

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Let's do some logic!

No HW

No Quizlet

Core Rules of Natural Deduction for TFL

Rules that rely on previous lines in the *main* proof

Reit	Reiteration
$\wedge$**I**	Conjunction Introduction
$\wedge$**E**	Conjunction Elimination
$\vee$**I**	Disjunction Introduction
$\rightarrow$**E**	Conditional Elimination
$\leftrightarrow$**E**	Biconditional Elimination

Rules that *also* (generally) occur *within* or care *about* sub-proofs

$\rightarrow$**I**	Conditional Introduction
$\leftrightarrow$**I**	Biconditional Introduction
$\vee$**E**	Disjunction Elimination
$\neg$**E**	Negation Elimination
$\neg$**I**	Negation Introduction
IP	Indirect Proof
Ex	Explosion

Rules that rely on previous lines in the *main* proof		Rules that *also* (generally) occur *within* or care *about* sub-proofs	
Reit	Reiteration	→I	Conditional Introduction
∧I	Conjunction Introduction	↔I	Biconditional Introduction
∧E	Conjunction Elimination	∨E	Disjunction Elimination
∨I	Disjunction Introduction	¬E	Negation Elimination
→E	Conditional Elimination	¬I	Negation Introduction
↔E	Biconditional Elimination	**IP**	Indirect Proof
DNE	Double-Negation Elim	**Ex**	Explosion
DS	Disjunctive Syllogism	**LEM**	Law of the Excluded Middle
MT	Modus Tollens		
DeM	DeMorgan Laws		

$E \vee F, F \vee G, \neg F \therefore E \wedge G$

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1	$E \vee F$	Pr.
2	$F \vee G$	Pr.
3	$\neg F$	Pr.
WTS:	$E \wedge G$	
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5	G	DS, 2, 3
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Recall: We invented a new symbol for us to use, in English, to talk about logic, the double turnstile: "$\models$"

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Reminder: Meaning of the Double Turnstile "$\vDash$"

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- "The sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$' together ENTAIL the sentence '$\mathscr{P}$'."
- "There's NO POSSIBLE CASE where all the sentences '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', are true and the sentence '$\mathscr{P}$' is false."

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We can say: "$\mathscr{A}, \mathscr{B}, \mathscr{C} \vDash \mathscr{P}$"

Some Proof-theoretic concepts!

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We can say: "$\mathscr{A}, \mathscr{B}, \mathscr{C} \vdash \mathscr{P}$"

Some Proof-theoretic concepts!

The Double Turnstile, "$\vDash$", is about *the existence of Valuations*.

I.e. it's about what sentences can be true and false at the same time (in, e.g., a given row of a truth-table).

The Single Turnstile, "$\vdash$", is about *the existence of Proofs*.

I.e. it's about whether there's a way to construct a formal proof that has those sentences located at specific lines.

Things we can define using '$\vdash$'

When a sentence is a "THEOREM of TFL"

To say that "Sentence '$\mathscr{A}$' is a THEOREM of TFL" is just to say:

Things we can define using '⊢'

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When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{and} \qquad \mathscr{B} \vdash \mathscr{A}$$

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When two sentences are "Provably Equivalent"

To say that "Sentence '$\mathscr{A}$' is Provably Equivalent to sentence '$\mathscr{B}$'," is just to say:

$$\mathscr{A} \vdash \mathscr{B} \qquad \textbf{\textit{and}} \qquad \mathscr{B} \vdash \mathscr{A}$$

This means that $\mathscr{A}$ can be proved from $\mathscr{B}$, ***and*** **vice versa**!

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row in that sentence's truth-table is a "F"

Logically Equivalent: These sentences always have the *same* truth-value for each row of their shared truth-table.

Logically Inconsistent: There's no row of their shared truth table where both sentences are true.

Valid$_{Semantic}$: In *ALL* the rows where every premise is assigned a "T", the conclusion is also assigned a "T".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction$_{Proof\text{-}Theoretic}$: You can prove the sentence's *negation* from no premises.

Provably Equivalent: You can prove one sentence using the other as your premise, *and* vice versa.

Provably Inconsistent: You can prove a contradiction using those sentences as two premises.

Valid$_{Proof\text{-}Theoretic}$: There exists a proof from those premises to that conclusion.

Semantic Concepts:

Tautology: Every row in that sentence's truth-table is a "T".

Contradiction: Every row of that sentence's *negation*'s truth table is "T".

Logically Equivalent: The biconditional between those two sentences is a *tautology*

Logically Inconsistent: The conjunction of those two sentences is a *contradiction*.

Valid$_{Semantic}$**:** There is ***NO*** row where all the premises are assigned "T" and the conclusion is assigned an "F".

Proof-Theoretic Concepts:

Theorem: There's a proof of the sentence from no premises.

Contradiction$_{Proof\text{-}Theoretic}$**:** You can prove the sentence's *negation* from no premises.

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Semantic Concepts:	Proof-Theoretic Concepts:
Tautology: $\vDash A$	**Theorem:** $\vdash A$
Contradiction: $\vDash \neg A$	**Contradiction**$_{\textit{Proof-Theoretic}}$**:** $\vdash \neg A$
Logically Equivalent: $\vDash A \leftrightarrow B$, (or, equivalently: "$A \vDash B$ and $B \vDash A$")	**Provably Equivalent:** $A \vdash B$ and $A \vdash B$ (or, equivalently: "$\vdash A \leftrightarrow B$")
Logically Inconsistent: $\vDash \neg(A \wedge B)$	**Provably Inconsistent:** $\vdash \neg(A \wedge B)$
Valid$_{\textit{Semantic}}$**:** $P_1, P_2, \ldots \vDash C$	**Valid**$_{\textit{Proof-Theoretic}}$**:** $P_1, P_2, \ldots \vdash C$

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NOTE: It is not a guarantee that these notions would line up. The proof theory and the semantics are completely detached from each other, and so we shouldn't expect this question to have an easy, simplistic answer.

TWO interesting questions:

Is Truth-Functional Logic *Sound*?

* Are all $\text{VALID}_{\text{Proof-Theoretic}}$ arguments also $\text{VALID}_{\text{Semantic}}$?

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* Are all sentences that are THEOREMS of TFL also TAUTOLOGIES?

That is, is every sentence that we can (from no premises) construct a formal proof of *also* a sentence whose truth-table has only “T”’s in every row under the main connective?

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* Are all sentences that are TAUTOLOGIES also THEOREMS of TFL?

That is, are all sentences whose truth-tables have only "T"'s in every row under the main connective also sentences that we can formally prove (from no premises)?

Proof Sketch: TFL is ***Sound***

We want to show that: "If $\mathscr{P} \vdash \mathscr{C}$, then $\mathscr{P} \vDash \mathscr{C}$"

* We show this via an *informal* proof, since we're making a proof *about* logic, but not doing logic (we're doing *meta*-logic!)
* We will use the fact that we defined a "sentence in TFL" using an "*inductive*" definition.

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  Recall: we defined sentences by saying:

  first, that the atomic letters are sentences of TFL

  second, that anything you produce by correctly applying a connective to a (pair of) sentence(s) of TFL produces a new sentence of TFL.
* We have to show this for all sentences of TFL, which would take forever if we tried to do them each individually.
* But we can duplicate the "*inductive*" method which we used to **define** a "sentence of TFL" for our proof..

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That is: We can show that TFL is sound by first showing that a small set of proofs are legit:

Step 1-a: Consider the "base-class" of small, one-line proofs which correspond to exactly one use of a rule, e.g.:
$\mathscr{A} \vdash \mathscr{A}$, $\mathscr{A}, \mathscr{B} \vdash \mathscr{A} \wedge \mathscr{B}$, $\mathscr{A} \wedge \mathscr{B} \vdash \mathscr{A}$, $\mathscr{A} \vdash \mathscr{A} \vee \mathscr{B}$,
$\mathscr{A} \rightarrow \mathscr{B}, \mathscr{A} \vdash \mathscr{B}$, etc.

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Step 1-b: Show that each of the possible proofs in the "base-class" are also ones whose truth-tables have no counterexamples. That is, take the proof $\mathscr{A} \vdash \mathscr{A} \vee \mathscr{B}$, and then show that
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Step 2: The next step is to show that **adding a line** to any proof that's $\text{V}{\small\text{ALID}}_{\vDash}$ into one that's **$\text{I}{\small\text{NVALID}}_{\vDash}$**.

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How to do it: Since adding a line is just a matter of **applying a rule**, what we need to do is show that, no matter what the proof is like, if it's already VALID$_{\vDash}$, then adding a line by applying the rule won't make it **INVALID**$_{\vDash}$.

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For example: We can argue that, if we have a VALID$_{\vDash}$ proof already, then adding a line using the rule "$\wedge I$" won't make it **INVALID**$_{\vDash}$ because applying the rule "$\wedge I$" will produce a sentence $\mathscr{C} \wedge \mathscr{D}$ where $\mathscr{C}$ and $\mathscr{D}$ are two previous lines the the proof.

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But, then, if $\mathscr{C}$ and $\mathscr{D}$ are two previous lines the the proof, then the truth-table will just be an expansion of the truth-table for $\mathscr{A}, \mathscr{B} \vdash \mathscr{A} \wedge \mathscr{B}$, which we've already shown is VALID$_{\vDash}$