Typicality-Based Chance

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Typicality-Based Chance: to be a chance is to be typically approximated by possible frequencies.

Typicality

Typical Property
Property P is typical in set Γ (relative to measure M) if and only if $M(\Gamma_P)$ is very close to $M(\Gamma)$.

- Γ_P: set of all elements in Γ which have P.
- M: finite measure over an algebra of sets which contains Γ_P.

Typical Property
Object x in set Γ is typical with respect to property P (relative to measure M) if and only if

(i) P is typical in Γ (relative to M), and
(ii) x has P.

The Functional Characterization of Chance

Typicality-Based Chance
For p_E to be the chance of E just is for it to be the case that for all ϵ greater than 0,

$$\lim_{n\to\infty} M_E\left(\left\{\omega \in \Omega_E \,\middle|\, \left|\frac{1}{n}\sum_{i=1}^{n} X_i^E(\omega) - p_E\right| < \epsilon\right\}\right) = 1 \qquad \text{(LLN)}$$

- E: event.
- Ω_E: set of all infinite sequences whose members are E and $\neg E$.
- For each i, X_i^E: random variable encoding whether, given ω in Ω_E, the ith place of ω is E or $\neg E$.
- M_E: finite measure defined over algebra of subsets in Ω_E.
- p_E: the chance of E obtaining.

Justifying the Measure

The Justification Question: for a given event E, what justifies a specific choice of the typicality measure $\mathrm{M_E}$ which figures in Typicality-Based Chance?

Here is how to answer this question in a given, particular case. Let c be a coin, and let $\mathrm{E_c}$ be the event of c landing heads after being flipped. Suppose we do not know the chance $\mathrm{p_{E_c}}$ of $\mathrm{E_c}$.

Make some reasonable assumptions about how to model, mathematically, the physical system in which c is flipped many times. Choose experimental error parameter $\epsilon = \frac{1}{20}$; note ϵ will be used to quantify closeness between the actual observed frequency of $\mathrm{E_c}$ and future guesses of the chance of $\mathrm{E_c}$. Let $\Omega_{\mathrm{E_c}}$ be the set of all possible infinite sequences of flips for c. And for each i, $\mathrm{X_i^{E_c}}$ says whether the ith flip lands heads.

The Goal: figure out both (i) the typicality measure $\mathrm{M_{E_c}}$, and (ii) the chance $\mathrm{p_{E_c}}$ of $\mathrm{E_c}$.

Towards that end, perform 1000 flips of c, and record the outcome of each flip. Let α_{1000} be the finite sequence encoding the outcomes, and let α be whichever sequence in $\Omega_{\mathrm{E_c}}$ is actual – so α is the physical possibility corresponding to the actual world.[1] Suppose that in total, c lands heads 519 times and tails 481 times. So the relative frequency of heads, in this first series of experiments, is $\frac{1}{1000}\sum_{\mathrm{i=1}}^{1000}\mathrm{X_i^{E_c}}(\alpha) = \frac{519}{1000} = .519$.

Now make the 'Unspecial Assumption': the observed frequency is typical. More precisely, the unspecial assumption is that whatever the right typicality measure $\mathrm{M_{E_c}}$ might be, and whatever the right chance might be,

(i) the property $\mathrm{P}^{.05}_{1000}$ of having a relative frequency of heads—in the first 1000 flips—which is within $\epsilon = .05$ of $\mathrm{p_{E_c}}$, is typical in $\Omega_{\mathrm{E_c}}$ (relative to $\mathrm{M_{E_c}}$), and

(ii) α is typical with respect to $\mathrm{P}^{.05}_{1000}$ (relative to $\mathrm{M_{E_c}}$).

A natural measure which respects both (i) and (i): the measure $\mathrm{M_{E_c}}$ which, for each i, assigns a size of $\frac{1}{2}$ to every set of possible infinite sequences in which the coin lands heads on the ith flip. We can check that this is, indeed, the right measure, by performing more flips, and seeing if the quantity

$$\mathrm{M_{E_c}}\left(\left\{\omega \in \Omega_{\mathrm{E_c}} \,\middle|\, \left|\frac{1}{\mathrm{n}}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{X_i^{E_c}}(\omega) - \frac{1}{2}\right| < .05\right\}\right)$$

gets closer and closer to 1 as the number of flips n grows from 1000 to 2000, to 3000, and so on – as the law of large numbers requires.

[1] Here we make several simplifying assumptions. For instance, we make the assumption, common in mathematical modeling, that c is flipped infinitely many times in the actual world.